COSC 5P05 - Introduction to Lambda-Calculus

Term Test 3

Question 1 (10 marks): Show that $[\![(\lambda z^{\sigma\times\sigma}.\mathrm{fst}(z)\ \langle x,x\rangle)]\!]_{\{x^\sigma\}} = p_x$ where $p_x : 1\times[\![\sigma]\!] \to [\![\sigma]\!]$.

Solution: In the following computation we use the projections $p_1 : [\![\sigma]\!]\times[\![\sigma]\!] \to [\![\sigma]\!]$ and $p_z : 1\times[\![\sigma]\!]\times([\![\sigma]\!]\times[\![\sigma]\!]) \to [\![\sigma]\!]\times[\![\sigma]\!]$.

$$\begin{aligned}
&[\![(\lambda z^{\sigma\times\sigma}.\mathrm{fst}(z)\ \langle x,x\rangle)]\!]_{\{x^\sigma\}}\\
&= \mathrm{eval}\circ\langle [\![\lambda z^{\sigma\times\sigma}.\mathrm{fst}(z)]\!]_{\{x^\sigma\}}, [\![\langle x,x\rangle]\!]_{\{x^\sigma\}}\rangle\\
&= \mathrm{eval}\circ\langle \Lambda([\![\mathrm{fst}(z)]\!]_{\{x^\sigma,z^{\sigma\times\sigma}\}}), \langle [\![x]\!]_{\{x^\sigma\}}, [\![x]\!]_{\{x^\sigma\}}\rangle\rangle\\
&= \mathrm{eval}\circ\langle \Lambda(p_1\circ[\![z]\!]_{\{x^\sigma,z^{\sigma\times\sigma}\}}), \langle p_x,p_x\rangle\rangle\\
&= \mathrm{eval}\circ\langle \Lambda(p_1\circ p_z), \langle p_x,p_x\rangle\rangle\\
&= \mathrm{eval}\circ(\Lambda(p_1\circ p_z)\times \mathrm{id}_{[\![\sigma]\!]\times[\![\sigma]\!]})\circ\langle \mathrm{id}_{1\times[\![\sigma]\!]}, \langle p_x,p_x\rangle\rangle\\
&= p_1\circ p_z\circ\langle \mathrm{id}_{1\times[\![\sigma]\!]}, \langle p_x,p_x\rangle\rangle\\
&= p_1\circ\langle p_x,p_x\rangle\\
&= p_x.
\end{aligned}$$

Question 2 (10 marks): Show that $\langle \mathrm{fst}(M),\mathrm{snd}(M)\rangle)]\!]_\Delta = [\![M]\!]_\Delta$ for every λ-term M.
Hint: Recall that $\langle f\circ h, g\circ h\rangle = \langle f,g\rangle\circ h$ and that $\mathrm{id}\times\mathrm{id} = \mathrm{id}$.

Solution:

$$\begin{aligned}
[\![\langle \mathrm{fst}(M), \mathrm{snd}(M)\rangle]\!]_\Delta &= \langle [\![\mathrm{fst}(M)]\!]_\Delta, [\![\mathrm{snd}(M)]\!]_\Delta \rangle \\
&= \langle p_1 \circ [\![M]\!]_\Delta, p_2 \circ [\![M]\!]_\Delta \rangle \\
&= \langle p_1, p_2 \rangle \circ [\![M]\!]_\Delta \\
&= (\mathrm{id} \times \mathrm{id}) \circ [\![M]\!]_\Delta \\
&= [\![M]\!]_\Delta.
\end{aligned}$$