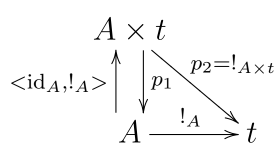
COSC 5P05 - Introduction to Lambda-Calculus

Term Test 2

Question 1 (10 marks): Let $\mathbb{C}$ be a cartesian category, i.e., a category with a terminal object t and products. Show that $A \times t \cong A$ for all objects A of $\mathbb{C}$.
Hint: Notice that $p_2 = !_{A\times t} = !_A \circ p_1$ *because* t *is a terminal object.*

Solution: Consider the following diagram:



We have

$$\begin{aligned} p_1 \circ < \mathrm{id}_A, !_A > &= \mathrm{id}_A, \\ < \mathrm{id}_A, !_A > \circ\, p_1 &= < p_1, !_A \circ p_1 > \\ &= < p_1, !_{A\times t} > && t \text{ terminal} \\ &= < p_1, p_2 > && t \text{ terminal} \\ &= \mathrm{id}_{A\times t}. \end{aligned}$$

Question 2 (10 marks): An ordered set $(A, \sqsubseteq)$ is a set A with an order relation, i.e., a relation that satisfies (reflexivity) $x \sqsubseteq x$, (transitivity) $x \sqsubseteq y$ and $y \sqsubseteq z$ implies $x \sqsubseteq z$, and (antisymmetry) $x \sqsubseteq y$ and $y \sqsubseteq x$ implies $x = y$. A monotonic function between two order sets $(A, \sqsubseteq)$ and $(B, \sqsubseteq)$ is a function that satisfies: if $x \sqsubseteq y$, then $f(x) \sqsubseteq f(y)$ for all $x, y \in A$.

1. Show that the structure Ord with ordered sets as objects and monotonic functions as morphisms is a category.

2. Show that Ord has products.

Hint: For 2. it is sufficient to show that the set of pairs can be ordered, that the projections are monotonic, and that the function $< f, g > (x) = (f(x), g(x))$ *is monotonic if* f *and* g *are.*

Solution:

1. It is sufficient to show that the identity function is monotonic and that composition of monotonic functions is also monotonic. The first property is trivial. From $x \sqsubseteq y$ we conclude $f(x) \sqsubseteq f(y)$ if f is monotonic. This implies for every monotonic g that $g(f(x)) \sqsubseteq g(f(y))$. This shows that $g \circ f$ is monotonic.

2. First, we show that the set of pairs $A \times B$ is an ordered set with order $(x, u) \sqsubseteq (y, v)$ iff $x \sqsubseteq y$ and $u \sqsubseteq v$. This relation is reflexive since $x \sqsubseteq x$ and $u \sqsubseteq u$ implies $(x, u) \sqsubseteq (x, u)$. It is transitive because $(x, u) \sqsubseteq (y, v)$ and $(y, v) \sqsubseteq (z, w)$ implies $x \sqsubseteq y$, $y \sqsubseteq z$, $u \sqsubseteq v$ and $v \sqsubseteq w$. Since A and B are ordered sets we obtain $x \sqsubseteq z$ and $u \sqsubseteq w$, showing that $(x, u) \sqsubseteq (z, w)$. Finally, the relation is also antisymmetric because $(x, u) \sqsubseteq (y, v)$ and $(y, v) \sqsubseteq (x, u)$ implies $x \sqsubseteq y$, $y \sqsubseteq x$, $u \sqsubseteq v$ and $v \sqsubseteq u$. Since A and B are ordered sets we obtain $x = y$ and $u = v$, showing that $(x, u) = (y, v)$. Now, suppose $(x, u) \sqsubseteq (y, v)$. Then we have $x \sqsubseteq y$ and $u \sqsubseteq v$, i.e., $p_1(x, u) \sqsubseteq p_1(y, v)$ and $p_2(x, u) \sqsubseteq p_2(y, v)$. Last but not least, suppose $x \sqsubseteq y$ and f and g are monotonic. Then we get $f(x) \sqsubseteq f(y)$ and $g(x) \sqsubseteq g(y)$ so that $(f(x), g(x)) \sqsubseteq (f(y), g(y))$ follows. This shows $< f, g > (x) \sqsubseteq < f, g > (y)$.