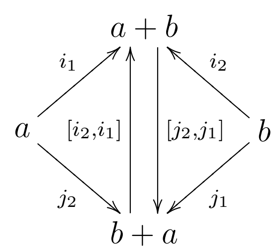
# COSC 5P05 - Introduction to Lambda-Calculus

# Term Test 2

**Question 1 (10 marks):** Let $\mathbb{C}$ be a category with coproducts. Show that $a + b \cong b + a$ for each pair of objects $a, b$ from $\mathbb{C}$.

**Solution:** Consider the following diagram:




We have

$$
\begin{aligned}
[i_2, i_1] \circ [j_2, j_1] \circ i_1 &= [i_2, i_1] \circ j_2 \\
&= i_1 \\
&= \mathrm{id}_{a+b} \circ i_1, \\
[i_2, i_1] \circ [j_2, j_1] \circ i_2 &= [i_2, i_1] \circ j_1 \\
&= i_2 \\
&= \mathrm{id}_{a+b} \circ i_2
\end{aligned}
$$

so that from the uniqueness of the coproduct morphism $[i_2, i_1] \circ [j_2, j_1] = \mathrm{id}_{a+b}$ follows. The equation $[j_2, j_1] \circ [i_2, i_1] = \mathrm{id}_{b+a}$ can be shown analogously.

**Question 2 (10 marks):** Let $\mathbb{C}_1$ and $\mathbb{C}_2$ be categories. Show that one can define a category $\mathbb{C}_1 \times \mathbb{C}_2$ whose objects are pairs $(a, b)$ of objects $a$ from $\mathbb{C}_1$ and $b$ from $\mathbb{C}_2$ and whose morphisms are pairs of morphisms from $\mathbb{C}_1$ and $\mathbb{C}_2$, that is $F \in \mathbb{C}_1 \times \mathbb{C}_2[(a, b), (c, d)]$ iff $F = (f, g)$ with $f \in \mathbb{C}_1[a, c]$ and $g \in \mathbb{C}_2[b, d]$.

**Solution:** Suppose $F = (f, g)$ and $G = (h, k)$ with $f \in \mathbb{C}_1[a, c]$, $g \in \mathbb{C}_2[b, d]$, $h \in \mathbb{C}_1[c, e]$ and $k \in \mathbb{C}_2[d, f]$. Then define $G \circ F = (h \circ f, k \circ g)$. Then we have

$$\begin{aligned}
F \circ (\mathrm{id}_a, \mathrm{id}_b) &= (f \circ \mathrm{id}_a, g \circ \mathrm{id}_b) \\
&= (f, g) \\
&= F, \\
(\mathrm{id}_c, \mathrm{id}_d) \circ F &= (\mathrm{id}_c \circ f, \mathrm{id} \circ g) \\
&= (f, g) \\
&= F,
\end{aligned}$$

i.e., $(\mathrm{id}_a, \mathrm{id}_b)$ is the identity on $(a, b)$. Associativity is shown as follows

$$\begin{aligned}
(H \circ G) \circ F &= ((l, m) \circ (h, k)) \circ (f, g) \\
&= (l \circ h, m \circ k) \circ (f, g) \\
&= (l \circ h \circ g, m \circ k \circ g), \\
H \circ (G \circ F) &= (l, m) \circ ((h, k) \circ (f, g)) \\
&= (l, m) \circ (h \circ f, k \circ g) \\
&= (l \circ h \circ g, m \circ k \circ g).
\end{aligned}$$

The last lines of each computation above is correct because composition in $\mathbb{C}_1$ and $\mathbb{C}_2$ is associative so that no brackets are needed.