A note on proximity spaces and connection based mereology

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Abstract: Representation theorems for systems of regions have been of interest for some time, and various contexts have been used for this purpose: Mormann [17] has demonstrated the fruitfulness of the methods of continuous lattices to obtain a topological representation theorem for his formalisation of Whiteheadian ontological theory of space; similar results have been obtained by Roeper [20]. In this note, we prove a topological representation theorem for a connection based class of systems, using methods and tools from the theory of proximity spaces. The key novelty is a new proximity semantics for connection relations.

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1. Introduction

Following de Laguna [5], Whitehead [27] developed a theory of space known as *pointless approach to geometry*. Whiteheadian theory is based on the primitive notion of spatial region and a binary relation C between regions, called *connection relation*. The

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notion of "point", the basic primitive notion of classical geometry, is now (second-order) definable in various ways as a special collection of regions; this way it becomes one of the very complex notions of the theory. We will elaborate briefly on this in Section 2, and refer the reader to the paper by Gerla [10] for a survey on pointless geometry.

The fact that relations such as "part-of", "overlap", "non-tangential inclusion" and others can be defined in terms of the connection relation relates some pointless geometries to the field of mereology [16] and to its fusion with topology – mereotopology [26]. The latter is closely related to *naive* or *qualitative physics*, introduced by [12], in particular, to its subfield *Qualitative Spatial Reasoning* (QSR) [3]. It has been realised that searching for models of mereological systems, methods of lattice theory and the theory of relation algebras can be fruitfully employed, see e.g. [22; 21; 8; 7]. It is remarkable, however, that in the fast growing field of mereology and mereotopology little attention has been paid to the theory of proximity spaces, the only exception being the early paper by [23], in which the notion of proximity distributive lattice was introduced, and a topological representation theorem was proved. It was mentioned there, that the paper can be considered as an attempt to build a pointless analogue to the notion of proximity space. The paper had been written under the direction of the Russian Professor V. A. Efremovic, the founder of the theory of proximity spaces [9]. This makes it clear that the possibility to build a pointless analogue to proximity theory, and in general, to topology, had been known to its originators.

Roughly speaking, a proximity space is a non-empty set X with a binary relation δ between subsets, called *proximity*, with the intuitive meaning that $A\delta B$ holds, when "A is near B" in some sense. The proximity relation satisfies axioms which are identical with some of the typical axioms of the connection relation. Each proximity space determines a natural topology with nice properties, and the theory possesses deep results, rich machinery and tools; the main work on proximity spaces is the book by S. A. Naimpally and B. D. Warrack [18].

In this paper we will apply the theory of proximity spaces to a concrete mereological system, based on a connection relation, called *connection algebra*. Natural examples of connection algebras are Boolean algebras of (closed) regular subsets of completely regular topological spaces; it is well known that these topologies correspond to proximity spaces. We prove that each connection algebra can be isomorphically embedded into the algebra of closed regions of a certain proximity space. Using the fact that each proximity space can be isomorphically embedded into a dense subset of a compact Hausdorff space by Smirnov's compactification theorem, we obtain a representation theorem for connection algebras with the classical standard topological interpretation of the connection relation, namely,

xCy iff the the (closed) regions x and y share a common point.

Other forms of connection have been studied by A. G. Cohn and A. Varzi [4]. Applications of proximity spaces to similar problems can be found in [25] and [6]. There, proximity spaces are used to formalise the notions of local and global similarity relations. A local similarity relation has a semantics just as the overlapping relation in mereology, and a global similarity relation is interpreted just by the proximity relation.

This shows another possible approach to the theory of mereological relations – the theory of similarity relations (or, more generally, informational relations) in information systems (see [24] for references). This, however, will be subject of future research.

The paper is organised as follows. In Section 1 we introduce the notion of connection algebra, as a Boolean algebra with an additional binary relation C satisfying a number of additional axioms. In Section 2 we give some definitions and facts from the theory of proximity spaces. Here we introduce the notion of *proximity connection algebra* as the algebra of closed regular subsets of a proximity space; the connection relation between regions is just the proximity relation. We prove here that each proximity connection algebra is isomorphic to a standard connection algebra in which closed regions are connected if and only if they share common point. In Section 3 we introduce the notion of cluster, which is an analogue of point just as maximal filters may be identified with points in the theory of Boolean algebras. Clusters in the theory of connection algebras are lattice-theoretical analogues of clusters in the theory of proximity spaces. The main result of the paper is a representation theorem for connection algebras: Every connection algebra can be isomorphically embedded into a proximity connection algebra and into standard connection algebra. The representation construction simulates the corresponding proof of Smirnov's compactification theorem for proximity spaces, using clusters. In Section 4 we discuss the new proximity semantics for the connection relation, some related works, and mention some open problems.

If X is a set, we denote by 2^X the powerset of X. If $Y \subseteq X$, we use just $-Y$ to denote the set complement of Y in X. The base set X will always be clear from the context. It should also be mentioned that we regard a relation R on a set X as a subset of $X \times X$. Thus, $x(-R)y$ has the same meaning as $\neg\, xRy$. For a topology τ on X we denote its closure operator by cl_τ, and its interior operator by int_τ. If no confusion can arise, we just write cl or int. Recall that cl and int are inter-definable by $\mathrm{cl}(Y) = -\mathrm{int}(-Y)$. We invite the reader to consult [13] or [15] for undefined notions in topology, and [14] for Boolean algebras.

Many of the results below are translations of facts about proximity spaces of which proofs are available in [18]. Since space is at a premium, we will usually cite this source for those proofs which are known and not immediately obvious.

2. Connection algebras

An algebraic system $B = (B, 0, 1, \vee, \wedge, *, C)$ is called a *connection algebra* (CA), if $(B, 0, 1, \vee, \wedge, ^*)$ is a Boolean algebra and C is a binary relation on B, satisfying the following axioms:

(1) If $x \wedge y \neq 0$ then xCy.
(2) If xCy then $x, y \neq 0$.
(3) xCy implies yCx.
(4) $xC(y \vee z)$ iff xCy or xCz.
(5) If $x(-C)y$ then $x(-C)z$ and $y(-C)z^*$ for some $z \in B$.

(6) If $x \not\leq y$ then zCx and $z(-C)y$ for some $z \in B$.

Usually, we shall just write (B,C) for a connection algebra. The elements of B will be called (spatial) *regions* and the relation C the *connection relation*. Note that our definition of CA differs from that of [21]. Axiom C6 is an extensionality axiom, since it implies in the presence of the other axioms that

$$x = y \Longleftrightarrow (\forall z \in B)[zCx \Longleftrightarrow zCy].$$

We define *non-tangential inclusion* $\ll$ by

$$x \ll y \Longleftrightarrow x(-C)y^* \tag{2.1}$$

This relation has different names in the literature: "well-inside relation", "well below", "interior parthood", or "deep inclusion". The relations C and $\ll$ are inter-definable, and the axiomatisation of C can be equivalently rewritten in terms of non-tangential inclusion. One possible axiomatisation is as follows:

(1) $1 \ll 1$,
(2) $x \ll y$ implies $x \leq y$.
(3) $x \leq y \ll z \leq t$ implies $x \ll t$.
(4) $x \ll y$ and $x \ll z$ implies $x \ll y \wedge z$.
(5) If $x \ll z$ then $x \ll y \ll z$ for some $y \in B$.
(6) $x \ll y$ implies $y^* \ll x^*$.
(7) If $x \not\leq y$ then $z \ll x$ and $z \not\ll y$ for some $z \in B$.

We use these equivalent axiomatisations of connection algebras, because they correspond to the axiomatic systems of proximity spaces given below. Note that axiom ($\ll$5) corresponds to axiom C5.

Our connection algebras are strongly related to the structures of [11] and [2]. All of our axioms, except C5, are either axioms or theorems in the Grzegorczyk system. Grzegorczyk assumes B to be a complete Boolean algebra but we have decided to drop this assumption in order to obtain a first-order notion of connection algebra. Grzegorczyk also assumes two non-trivial axioms in his system, containing the definable notion of point. In pointless geometry, the definition of *point* is one of the central problems, and it is defined (in different ways) as a collection of sets (or sequences) of regions. Including the notion of point in the set of axioms, as Grzegorczyk did, makes his system complicated and not first-order. [1] have discussed the problem of equivalence of the notion of point in the systems of Grzegorczyk and Whitehead. They proved that the two notions are equivalent if the Grzegorczyk system is enriched with axiom ($\ll$5) above. They also noted that this axiom is satisfied in the models of the connection relation in "good" topological spaces, for instance in all normal spaces, in particular, in n-dimensional Euclidean spaces. Therefore, we include axiom ($\ll$5) (or its equivalent form C5) in order to obtain a representation theorem such as the topological representation theorem in Grzegorczyk's system without the notion of point.

The following Lemma collects some easy properties of the connection relation:

LEMMA 2.1. *Suppose that (B,C) is a connection algebra.*

(1) If xCy and $x \leq x'$ and $y \leq y'$ then $x'Cy'$,
(2) xCx iff $x \neq 0$,
(3) $x \leq y$ iff $(\forall z)(zCx$ implies $zCy)$.

3. Proximity spaces and their topologies

In this Section we explore the basic definitions and properties of proximity spaces and their associated topologies. It will turn out that a connection algebra can be regarded as a special kind of proximity space.

If X is a nonempty set, then a binary relation δ on the powerset of X is called a *proximity*, if it satisfies C1 – C5 on the Boolean algebra $\langle X, \emptyset, X, \cup, \cap, -\rangle$; the pair $\langle X, \delta \rangle$ is called a *proximity space*. If $A, B \subseteq Y \subseteq X$, then $A\delta_Y B \iff A\delta B$ defines a proximity on Y which – with some abuse of notation – we will also denote by δ. Furthermore, instead of writing $\{x\}\delta\{y\}$, we will simply write $x\delta y$.

A proximity (space) is called *separated* if it satisfies

(6) $x\delta y$ implies $x = y$.

Defining $A \ll_\delta B$ iff $A(-\delta)(-B)$ we obtain the analogue of non-tangential inclusion. As with C and the non-tangential inclusion $\ll$, the relations δ and $\ll_\delta$ are interdefinable, and the definition of proximity space can be re-axiomatised in terms of $\ll_\delta$ by using ($\ll$1) – ($\ll$6).

Here are a few examples of proximity spaces:

(1) Let $A, B \subseteq X$, and set $A\delta B \iff A \neq \emptyset$ and $B \neq \emptyset$. This is the *trivial proximity on* X. Note that in general it does not satisfy C6.
(2) Let $\langle X, \tau \rangle$ be a normal topological space and define

(3.1) $$A\delta B \iff \mathrm{cl}A \cap \mathrm{cl}B \neq \emptyset.$$

We call this proximity the *standard proximity*. It is separated iff $\langle X, \tau \rangle$ is a Hausdorff space.
(3) Let $\langle X, \tau \rangle$ be a locally compact Hausdorff space, $A, B \subseteq X$, and define $A(-\delta)B$ iff $\mathrm{cl}A \cap \mathrm{cl}B = \emptyset$ and $\mathrm{cl}A$ or $\mathrm{cl}B$ are compact.
(4) Let $\langle X, d \rangle$ be a metric space, $A, B \subseteq X$, and define $A\delta B$ iff $d(A, B) = 0$, where $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$.
(5) Let $\langle X, \tau \rangle$ be a completely regular space. Two subsets A, B of X are *functionally distinguishable* iff there is a continuous real-valued function $f : X \to [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$. Then we can define a proximity δ in X by $A(-\delta)B$ iff A and B are functionally distinguishable.

Define an operator cl on 2^X by

(3.2) $$\mathrm{cl}(A) = \{x \in X : x\delta A\}.$$

Proofs of the following results can be found in [18]:

THEOREM 3.1. *(1) The operation of* (3.2) *defines the closure operator of a topology* $\tau(\delta)$ *on* X.

(2) $(X,\tau(\delta))$ *is a completely regular space. If* δ *is separated, then* $(X,\tau(\delta))$ *is Tihonoff space, i.e. completely regular and* T_1.

(3) $A\delta B$ *iff* $\mathrm{cl}(A)\,\delta\,\mathrm{cl}(B)$.

If (X,τ) *is a topological space, we say that* X admits a proximity*, if there is a proximity* δ *on* X *such that* $\tau=\tau(\delta)$.

(4) If (X,τ) *is a compact Hausdorff space, then it admits a unique proximity* δ*, defined by* $A\delta B$ *iff* $\mathrm{cl}(A)\cap\mathrm{cl}(B)\neq\emptyset$.

(5) $A\ll_\delta B$ *implies* $\mathrm{cl}(A)\ll_\delta B$ *and* $A\ll_\delta \mathrm{int}(B)$.

We say that a subset A of X is *regular closed* if $A=\mathrm{cl}(\mathrm{int}(A))$. Clearly, A is regular closed, if it is a closure of an open set B. The next result shows that the density axiom ($\ll$5) is witnessed by a regular closed set:

LEMMA 3.2. *If* $A\ll_\delta B$ *then there exists a regular closed set* C *such that* $A\ll_\delta C\ll_\delta B$.

PROOF. By ($\ll$5) there exists some D such that $A\ll D\ll B$. Applying Theorem 3.1.v we obtain

$$A\ll_\delta \mathrm{int}(D)\subseteq \mathrm{cl}(\mathrm{int}(D))\ll_\delta B,$$

and thus, $A\ll_\delta \mathrm{cl}(\mathrm{int}(D))\ll_\delta B$ by ($\ll$3). □

Next, we will consider connection algebras over proximity spaces. First, recall that for any topological space (X,τ), the collection $RC(X)$ of regular closed sets can be made into a complete Boolean algebra $(RC(X),0,1,\wedge,\vee,*)$, by setting

$$1=X,\ 0=\emptyset, A^*=\mathrm{cl}(-A),\ \ A\vee B=A\cup B,\ A\wedge B=(A^*\vee B^*)^*.$$

Then, $(RC(X),\delta)$ is a system of the same type as a connection algebra, where δ is inherited from (X,τ). Indeed, more is true:

LEMMA 3.3. *Let* (X,δ) *be a proximity space. Then,* $(RC(X),\delta)$ *is a connection algebra.*

PROOF. The verification of axioms C1 – C4 is straightforward. C5 (which is equivalent to ($\ll$5)) follows from Lemma 3.2. Finally, C6 follows from the fact that $(X,\tau(\delta))$ is (completely) regular (see Theorem 3.1,ii). □

$(RC(X),\delta)$ is called the *proximity connection algebra over* (X,δ). Following the definition in Example 2, we call it a *standard connection algebra*, if

$$A\delta B \text{ iff } A\cap B\neq\emptyset. \tag{3.3}$$

for all $A,B\in RC(X)$. The next result shows that it suffices to consider only standard connection algebras.

Let (X,δ) and (X',δ') be two proximity spaces. A one-one mapping f from X onto X' is called δ-*homeomorphism* if for any subsets A,B of X we have $A\delta B$ iff $f(A)\delta' f(B)$. It is known that f is also topological homeomorphism from the topological spaces $(X,\tau(\delta))$ onto $(X',\tau(\delta'))$ [18].

THEOREM 3.4. **(Isomorphism theorem)** *Each proximity connection algebra is isomorphic to a standard connection algebra.*

PROOF. Let $(RC(X),\delta)$ be the proximity connection algebra over (X,δ). By the Smirnov Compactification Theorem (see [18], Theorem (7.7)) there exist a compact Hausdorff space (Y,τ) and a δ-homomorphism f from X onto a dense subspace Y_0 of Y, such that for any $A,B\subseteq X$, $A\delta B$ iff $\mathrm{cl}_\tau(f(A))\cap \mathrm{cl}_\tau(f(B))\neq\emptyset$. Example 2 tells us that

$$P\delta_Y Q \Longleftrightarrow \mathrm{cl}_\tau(P)\cap \mathrm{cl}_\tau(Q)\neq\emptyset$$

defines a standard proximity on Y. Now let $(RC(Y_0),\delta_Y)$ be the proximity connection algebra over Y_0 inherited from (Y,δ_Y). Since f is also a homeomorphism, the connection algebra $(RC(X),\delta)$ is isomorphic to the corresponding connection algebra in the dense subspace Y_0 of Y. Now, observing that $f(A)$ is closed in τ_Y for any closed set A of τ, we obtain for all $A,B\in RC(X)$

$$A\delta B \Longleftrightarrow \mathrm{cl}_\tau(f(A)\cap \mathrm{cl}_\tau(f(B))\neq\emptyset \Longleftrightarrow f(A)\cap f(B)\neq\emptyset.$$

The equivalences above show that $(RC(Y_0),C_0)$ is standard, and that f is an isomorphism from $(RC(X),C)$ onto $(RC(Y_0),C_0)$. $\square$

4. A representation theorem for connection algebras

In this section we shall prove that each connection algebra can be isomorphically embedded into a proximity connection algebra and thus, into a standard proximity connection algebra. Our strategy follows the proof of the Stone representation theorem for Boolean algebras. In a Stone space $\mathrm{S}(B)$, points are maximal filters. In connection algebras, points of the representation space will be analogues of maximal filters, called *clusters*. We will take the notion of a cluster from the theory of proximity spaces, and our definition is just the lattice-theoretic translation of the corresponding definition of cluster from [18], Definition 5.4. Many statements for clusters in connection algebras have identical proofs (up to the aforementioned lattice-theoretical translation) as the proofs of the corresponding statements for clusters in proximity spaces. When such identical proofs exist we will refer to the corresponding statement and its proof given in [18].

Throughout this Section we suppose that (B,C) is a connection algebra.

A nonempty subset Γ of B is called a *cluster* if the following conditions are satisfied:

(1) If $x,y\in\Gamma$ then xCy.
(2) If xCy for every $y\in\Gamma$, then $x\in\Gamma$.
(3) If $x\vee y\in\Gamma$ then $x\in\Gamma$ or $y\in\Gamma$.

The set of all clusters of B is denoted by $\mathrm{Clust}(B)$. It is not hard to see that each maximal filter U of B is contained in exactly one cluster $m(U)$, and that the assignment $U \mapsto m(U)$ is an onto mapping from $\mathrm{S}(B)$ to $\mathrm{Clust}(B)$.

Our strategy is as follows: First, we define a suitable proximity δ_B on $\mathrm{Clust}(B)$. Then, we find a one-one Boolean homomorphism h which preserves C from B into the proximity connection algebra over $\langle \mathrm{Clust}(B), \delta_B \rangle$. Finally, we invoke Theorem 3.4 for an isomorphism from $\langle \mathrm{Clust}(B), \delta_B \rangle$ into a standard connection algebra.

The following properties of clusters will be helpful later:

LEMMA 4.1. *Let* $\Gamma \in \mathrm{Clust}(B)$, *and* $a, b \in B$.

(1) If $a \in \Gamma$ *and* $a \leq b$, *then* $b \in \Gamma$.
(2) If aCb, *then there is some* $\Delta \in \mathrm{Clust}(B)$ *such that* $a \in \Delta$ *and* $b \in \Delta$.
(3) $a^* \in \Gamma$ *iff for all* $b, c \in B$, $c \in \Gamma$ *and* $b \vee a = 1$ *imply* cCb.

PROOF. i. Suppose that $a \in \Gamma$ and $a \leq b$. Let $d \in \Gamma$; then, aCd by CL1, and it follows from Lemma 2.1.i that dCb. Hence, $b \in \Gamma$ by CL2.

ii. [18], Theorem 5.13.

iii. If $a^*, c \in \Gamma$ and $a \vee b = 1$, then $a^* \leq b$. It follows from i. that $b \in \Gamma$, and hence, cCb by CL1.

Conversely, suppose that for all $b, c \in B$, $c \in \Gamma$ and $b \vee a = 1$ imply cCb. Setting $b = a^*$, we obtain that cCa^* for all $c \in \Gamma$, and thus, $a^* \in \Gamma$ by CL2. □

We define a function $h : B \to \mathrm{Clust}(B)$ by

$$h(a) = \{\Gamma \in \mathrm{Clust}(B) : a \in \Gamma\} \tag{4.1}$$

in analogy to the Stone representation theorem for Boolean algebras. The following properties are easily proved from Lemma 4.1:

LEMMA 4.2. *(1)* $h(0) = \emptyset$, $h(a) = \mathrm{Clust}(B) \iff a = 1$.
(2) $h(a \vee b) = h(a) \cup h(b)$.
(3) aCb *iff* $h(a) \cap h(b) \neq \emptyset$.

Next, we set for $X, Y \subseteq \mathrm{Clust}(B)$

$$X \delta_B Y \text{ iff } (\forall x, y \in B)[x \in \bigcap X \text{ and } y \in \bigcap Y \text{ imply } xCy].$$

By definition of h, we have

$$X \delta_B Y \text{ iff } (\forall x, y \in B)[X \subseteq h(x) \text{ and } Y \subseteq h(y) \text{ imply } xCy].$$

Let $PS(B)$ be the structure $\langle \mathrm{Clust}(B), \delta_B \rangle$. The proof of the next result can be obtained by (part of) the proof of the Smirnov Compactification Theorem in [18], Lemma 7.2:

THEOREM 4.3. *$PS(B)$ is a separated proximity space.*

Let X be the powerset of $\mathrm{Clust}(B)$, and τ be the topology on X induced by δ_B. Then, for each $M \in X$ we have

$$\mathrm{cl}(M) = \{\Gamma \in \mathrm{Clust}(B) : (\forall x, y \in B)[x \in \Gamma \text{ and } M \subseteq h(y) \Longrightarrow xCy]\}$$

We are now ready to prove our main result:

THEOREM 4.4. **(Representation Theorem)**

(1) *Each connection algebra can be embedded into a proximity connection algebra.*

(2) *Each connection algebra can be embedded into a standard connection algebra.*

PROOF. Let (B, C) be a connection algebra and $h : B \to \mathrm{Clust}(B)$ be defined by (4.1). Our aim is to prove that h is a one-one Boolean homomorphism $B \to RC(X)$ such that $aCb \Longleftrightarrow h(a) \cap h(b) \neq \emptyset$. This will show that (B, C) is isomorphic to a substructure of the proximity connection algebra over $(\mathrm{Clust}(B), \delta_B)$, and thus, prove i. By Theorem 3.4, this algebra in turn is isomorphic to a standard connection algebra, which finishes the proof.

We first show that $h(a^*) = \mathrm{cl}(-h(a))$:

$$\begin{aligned}
\Gamma \in h(a^*) &\Longleftrightarrow a^* \in \Gamma \\
&\Longleftrightarrow (\forall b, c \in B)[c \in \Gamma \text{ and } b \vee a = 1 \Longrightarrow cCb], && \text{by Lemma 4.1.iii.} \\
&\Longleftrightarrow (\forall b, c \in B)[c \in \Gamma \text{ and } h(b \vee a) = \mathrm{Clust}(B) \Longrightarrow cCb], && \text{by Lemma 4.2.i.} \\
&\Longleftrightarrow (\forall b, c \in B)[c \in \Gamma \text{ and } h(b) \cup h(a) = \mathrm{Clust}(B) \Longrightarrow cCb], && \text{by Lemma 4.2.ii.} \\
&\Longleftrightarrow (\forall b, c \in B)[c \in \Gamma \text{ and } -h(a) \subseteq h(b) \Longrightarrow cCb], \\
&\Longleftrightarrow \Gamma \in \mathrm{cl}(-h(a)).
\end{aligned}$$

This implies that h is well defined, i.e. that $h(a) \in RC(X)$, since

$$h(a) = \mathrm{cl}(-h(a^*)) = \mathrm{cl}(-\mathrm{cl}(-h(a))) = \mathrm{cl}(\mathrm{int}(h(a))).$$

Furthermore, it shows that $h(a^*) = \mathrm{cl}(-h(a)) = h(a)^*$, so that h preserves complements. Together with Lemma 4.2, it follows that h is a Boolean homomorphism.

To show that h is one-one, suppose that $a \neq b$, and let w.l.o.g $a \not\leq b$. By C6, there is some $c \in B$ such that aCc and $b(-C)c$. Let $\Gamma \in \mathrm{Clust}(B)$ such that $a, c \in \Gamma$, which exists by Lemma 4.1.ii. It follows now from $b(-C)c$ and CL1 that $b \notin \Gamma$.

If aCb, then Lemma 4.1.ii tells us that $h(a) \cap h(b) \neq \emptyset$. Conversely, if $h(a) \cap h(b) \neq \emptyset$, then aCb by CL1. □

5. Concluding remarks

In this note, we have demonstrated the usefulness of the theory of proximity spaces in connection based mereology. The proximity definition of the connection relation between (closed) regions is a new semantics for this type of relation which has not been used so far. The standard meaning of the connection between regions is not always suitable to describe the spatial configuration between them. For instance, the closed regions in the topological space $\mathbb{Q}$ of rational numbers have the same spatial nature as those in the space $\mathbb{R}$ of real numbers. But in $\mathbb{Q}$, the closed regions $A = \{x : 1 \leq x^2 \leq 2\}$ and $B = \{x : 2 \leq x^2 \leq 4\}$ are not standardly connected, because they do not share a common point, while in $\mathbb{R}$ they are connected. This stems from the fact that $\mathbb{Q}$ does not have enough points. If we consider $\mathbb{Q}$ as a proximity space, defined by the natural metric δ in $\mathbb{Q}$ (see Example 4 of proximity spaces), then we have $A\delta B$, because $d(A,B) = 0$; thus, they are connected by the proximity definition of the connection relation. According to the Isomorphism Theorem, proximity connection algebras describe the same spatial picture as the corresponding standard ones which are obtained by the compactification construction of Smirnov's Theorem, in which new points are added. This shows that proximity semantics and the standard semantics for the connection relation are in a sense equivalent. For instance, by applying this theorem to the space $\mathbb{Q}$ of rational numbers, we obtain the space $\mathbb{R}$ of real numbers with the two infinities $+\infty$ and $-\infty$, which contains $\mathbb{Q}$ as a dense subspace.

The representation theorem for connection algebras, which we have presented in the paper, shows that the axioms indeed characterise the intended spatial properties of the regions. As we have mentioned, topological representation theorems have been obtained earlier by Roeper [20] and Mormann [17]. Both authors assume that the Boolean algebra of regions is complete and both obtain similar results – the region-based mereotopology is equivalent to the point-based topology of locally compact Hausdorff spaces. These spaces can be considered as compactifications of local proximity spaces, and it can be noted that the axioms of Roeper's system are almost identical to those for local proximity spaces [18]. Thus, the theory of proximity spaces can be applied to the systems of Mormann and Roeper as well to obtain their representation results.

We hope to extend our representation theorem to the region connection calculus (RCC) of [19], in which the connection relation satisfies C1 – C4, C6, and also

(7) If $x \neq 0, 1$, then xCx^*.

Note that the Boolean Connection Algebras (BCA) of [21] are just another formulation of RCC. There are two difficulties here. One, that BCAs do not satisfy the axiom C5, which is essential in the application of proximity spaces, and also plays a prominent role in slightly different versions in the representation theorems of Mormann and Roeper. Grzegorczyk [11] does not use C5, but to obtain a representable system he uses axioms containing the (second-order) notion of point. Our conjecture is that BCAs, and consequently RCC, are not topologically representable in the sense that they do not contain enough axioms in order to obtain a topological representation.

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