



Brock University

Department of Computer Science

A Discrete Representation of  
Dicomplemented Lattices

Ivo Düntsch, Léonard Kwuida, Ewa Orloska,

Brock University  
Department of Computer Science  
St. Catharines  
Ontario Canada L2S 3A1

Institute of Telecommunications  
Szachowa 1, 04-894  
Warszawa, Poland

Berner Fachhochschule  
Fachbereich Wirtschaft  
CH-3005, Bern, Switzerland

Technical Report # CS-17-01  
March 2017

---

# A discrete representation of dicomplemented lattices

Ivo Düntsch\*  
Dept of Computer Science  
Brock University  
St Catharines, ON, L2S 3A1, Canada  
duentsch@brocku.ca

Léonard Kwuida  
Bernern Fachhochschule  
Fachbereich Wirtschaft  
CH-3005, Bern, Switzerland  
leonard.kwuida@bfh.ch

Ewa Orłowska<sup>†</sup>  
Institute of Telecommunications  
Szachowa 1  
04–894, Warszawa, Poland  
orłowska@itl.waw.pl

## Abstract

Dicomplemented lattices were introduced as an abstraction of Wille’s concept algebras which provided negations to a concept lattice. We prove a discrete representation theorem for the class of dicomplemented lattices. The theorem is based on a topology free version of Urquhart’s representation of bounded general lattices.

## 1 Introduction

In philosophical logic a concept is characterized by its extent and intent. The extent of a concept  $C$  meaningful in a universe of discourse  $U$  is a subset  $e(C)$  of  $U$  consisting of those objects which are the instances of the concept. The intent of a concept is the set  $i(C)$  of properties which qualify the objects of  $e(C)$  as instances of the concept. In [13] the notion of Dedekind cuts used in the definition of real numbers was generalized with the intuition that, when considered on a lattice, the cuts represent concepts. More precisely, a cut is a pair  $\langle A, B \rangle$  of subsets of a lattice such that there is a concept  $C$  with  $e(C) = A$  and  $i(C) = B$ . Concept lattices were defined endowed with a negation and a representation theorem was proved. In [6] an abstract algebraic representation of concepts and their negations is presented within the framework of dicomplemented lattices. The lattices are endowed with two negations; details can be found in Section 5. When considered on a distributive lattice, they are the counterparts to the Heyting and Brouwer negation, respectively, see [11].

---

\*Ivo Düntsch gratefully acknowledges support by the Natural Sciences and Engineering Research Council of Canada Discovery Grant 250153 and by the Bulgarian National Fund of Science, contract DN02/15/19.12.2016.

<sup>†</sup>Ewa Orłowska gratefully acknowledges support from the National Science Centre project DEC-2011/02/A/HS1/00395.

Since the early 20th century a variety of logical systems whose languages included the operators turning propositions into the opposite propositions have been a subject of extensive studies. Helena Rasiowa contributed substantially to that line of research. In her book “An Algebraic Approach to non-Classical Logics” [10] a survey of classes of systems with negations and their classification is provided. Dicomplemented lattices are a valuable enlargement of that classification.

In the present paper we prove a discrete representation theorem for the class of dicomplemented lattices. The definition of representation algebra does not involve any topology and therefore it is referred to as a discrete representation. The theorem is based on a discrete version of Urquhart’s representation of bounded general lattices [12].

It is well known that in case of distributive lattices there are discrete representation theorems both for the lattices and the lattice frames, e.g. discrete versions of the Stone and Priestley dualities. In this paper we show that there are Urquhart general lattice frames (doubly ordered sets) which do not admit a representation theorem for frames.

## 2 First definitions and notation

A *two sorted frame* is a triple  $\langle X, Y, R \rangle$  where  $X, Y$  are nonempty sets and  $R \subseteq X \times Y$  is a binary relation among elements of  $X$  and  $Y$ . Two sorted frames are called *polarities* in [1] and *formal contexts* in [13]. If  $X = Y$ , we usually just write  $\langle X, R \rangle$  and speak of a frame. We let  $R^\smile$  be the relational converse of  $R$ , i.e.  $R^\smile = \{\langle y, x \rangle \in Y \times X : xRy\}$ . If  $x \in X$ , then  $R(x) = \{y \in Y : xRy\}$ ;  $R^\smile(y)$  is defined analogously.

Common operators on the powerset of frames are

$$\begin{aligned} \langle R \rangle(Z) &= \{x \in X : R(x) \cap Z \neq \emptyset\} && \text{Possibility operator,} \\ [R](Z) &= \{x \in X : R(x) \subseteq Z\} && \text{Necessity operator.} \end{aligned}$$

For useful facts about these operators we invite the reader to consult [9, Section 1.8]. In particular, we shall use

$$\langle R \rangle(Z) = -[R](-Z), \tag{2.1}$$

$$\langle R; S \rangle(Z) = \langle R \rangle \langle S \rangle(Z) \text{ and } [R; S](Z) = [R][S](Z). \tag{2.2}$$

If  $f : X \rightarrow Y$  is a function,  $\text{ran } f = \{f(x) : x \in X\}$  is the *range of  $f$* .

Throughout,  $\langle L, \cdot, +, 0, 1 \rangle$  is a bounded lattice, possibly with additional operators. The set of all proper filters, respectively, ideals, of  $L$  is denoted by  $\mathcal{F}$ , respectively, by  $\mathcal{I}$ . Furthermore,  $\uparrow M = \{a \in L : (\exists b)[b \in M \text{ and } b \leq a]\}$ ; the set  $\downarrow M$  is defined dually. If  $M = \{a\}$  we usually just write  $\uparrow a$  instead of  $\uparrow \{a\}$ . A subset  $M$  of  $L$  is called *join dense* if every nonzero element of  $L$  is a join of elements of  $M$ ; *meet density* is defined dually. The filter of  $L$  generated by  $M$  is denoted by  $[M]_{\text{filt}}$ .

A *closure operator* on a partially ordered set  $\langle P, \leq \rangle$  is a mapping  $f : P \rightarrow P$  such that for all  $a, b \in P$

1.  $a \leq f(a)$ , (Extensive)
2.  $a \leq b$  implies  $f(a) \leq f(b)$ , (Isotone)

$$3. f(f(a)) = f(a). \quad (\text{Idempotent})$$

$a \in P$  is called *closed* (with respect to  $f$ ) if  $a = f(a)$ . An interior operator on  $\langle P, \leq \rangle$  is defined dually. In the case of  $\langle 2^X, \subseteq \rangle$  the smallest closed element is  $f(\emptyset)$  and the largest closed element is  $X$ .

Let  $\langle P, \leq \rangle$  and  $\langle Q, \preceq \rangle$  be partially ordered sets. A *Galois connection* is a pair of functions  $f : P \rightarrow Q$ ,  $g : Q \rightarrow P$  such that for all  $a \in P, b \in Q$

$$b \preceq f(a) \iff a \leq g(b). \quad (2.3)$$

It is well known that both  $f$  and  $g$  are antitone (order reversing).

There are close connections among closure operators, Galois connections and complete lattices. In particular, each closure operator on some  $2^X$  induces a complete lattice; G. Birkhoff [1, p. 49] credits this result to E.H. Moore [7, pp 53–80]:

**Theorem 2.1.** [1, Ch IV, Theorem 1] *Let  $f$  be a closure operator on the ordered powerset  $\langle 2^X, \subseteq \rangle$  of some nonempty set  $X$ , and let  $C$  be the collection of closed subsets of  $X$ . Then,  $C$  can be made into a complete lattice with the operations*

$$\sum_{i \in I} A_i = f \left( \bigcup_{i \in I} A_i \right), \quad (2.4)$$

$$\prod_{i \in I} A_i = \bigcap_{i \in I} A_i. \quad (2.5)$$

**Theorem 2.2.** 1. [4, Theorem 4] *Let  $\langle P, \leq \rangle$  be a partially ordered set and  $c$  be a closure operator on  $P$ . Then, there are an ordered set  $\langle Q, \preceq \rangle$  and a Galois connection  $\langle f, g \rangle$  such that  $c(a) = g(f(a))$  for all  $a \in P$ .*

2. [8, Theorem 2] *If  $\langle f, g \rangle$  is a Galois connection between the partially ordered sets  $\langle P, \leq \rangle$  and  $\langle Q, \preceq \rangle$ , then  $g \circ f : P \rightarrow P$  is a closure operator.*

### 3 Polarities and concept lattices

A *sufficiency operator* on  $L$  is a function  $f : L \rightarrow L$  for which for all  $a, b \in L$ ,

$$f(0) = 1, \quad \text{Co-normal} \quad (3.1)$$

$$f(a + b) = f(a) \cdot f(b) \quad \text{Co-additive.} \quad (3.2)$$

A *dual sufficiency operator* is a function  $g : L \rightarrow L$  for which for all  $a, b \in L$ ,

$$g(1) = 0, \quad (3.3)$$

$$g(a \cdot b) = g(a) + g(b). \quad (3.4)$$

These operators were first considered in a logical setting in [5].

It is easy to see that sufficiency operators and dual sufficiency operators are antitone.

Define functions  $[[R^\sim]] : 2^X \rightarrow 2^Y$ ,  $[[R]] : 2^Y \rightarrow 2^X$  by

$$\begin{aligned} [[R^\sim]](V) &= \{y \in Y : V \subseteq R^\sim(y)\}, \\ &= \{y \in Y : (\forall x \in X)[x \in V \Rightarrow xRy]\} \end{aligned} \quad (3.5)$$

$$\begin{aligned} [[R]](W) &= \{x \in X : W \subseteq R(x)\}, \\ &= \{x \in X : (\forall y \in Y)[y \in W \Rightarrow xRy]\}. \end{aligned} \quad (3.6)$$

These mappings are complete sufficiency operators. They are called the *polars of*  $\langle X, Y, R \rangle$  by Birkhoff [1, p. 56], and *derivation operators* by Wille [13].

**Theorem 3.1.** [1, p. 54] *Let  $A \subseteq X, B \subseteq Y$ .*

1. *The pair  $\langle [[R^\sim]], [[R]] \rangle$  forms a Galois connection.*
2. *The correspondences  $A \mapsto [[R^\sim]](A)$  and  $B \mapsto [[R]](B)$  are complete dual isomorphisms between the complete lattices of  $[[R]][[R^\sim]]$  – closed subsets of  $2^X$  and  $[[R^\sim]][[R]]$  – closed subsets of  $2^Y$ .*

To simplify notation we shall write  $f_R$  for  $[[R]][[R^\sim]]$  and  $g_R$  for  $[[R^\sim]][[R]]$ . We call the complete lattice of  $f_R$  – closed subsets of  $2^X$  the *B – complex algebra of the polarity*  $\langle X, Y, R \rangle$ , and denote it by  $\mathfrak{Cm}_B \langle X, Y, R \rangle$ .

Birkhoff's construction is the basis of formal concept analysis introduced by Wille [13]. A *formal concept of*  $\langle X, Y, R \rangle$  is a pair  $\langle A, B \rangle \in 2^X \times 2^Y$  such that  $[[R^\sim]](A) = B$  and  $[[R]](B) = A$ . The set of all formal concepts of  $\langle X, Y, R \rangle$  is denoted by  $\mathfrak{Con} \langle X, Y, R \rangle$ .

**Theorem 3.2.** [13] *Let  $\langle X, Y, R \rangle$  be a frame.  $\mathfrak{Con} \langle X, Y, R \rangle$  is a complete lattice, called the concept lattice of  $\langle X, Y, R \rangle$  under the lattice ordering  $\langle A, B \rangle \leq \langle A', B' \rangle$  if and only if  $A \subseteq A'$ .*

*Conversely, a complete lattice  $M$  is isomorphic to  $\mathfrak{Con} \langle X, Y, R \rangle$  if and only if there are mappings  $\gamma : X \rightarrow M$ ,  $\mu : Y \rightarrow M$  such that  $\text{ran } \gamma$  is join dense,  $\text{ran } \mu$  is meet dense, and  $xRy$  if and only if  $\gamma(x) \leq \mu(y)$  for all  $x \in X, y \in Y$ .*

It is not hard to see that the lattice  $\mathfrak{Con} \langle X, Y, R \rangle$  is isomorphic to the lattice of  $f_R$  – closed sets of  $X$  and dually isomorphic to the  $g_R$  – closed sets of  $Y$ . Thus, the concept lattice of a frame is (isomorphic to) the complex algebra of the polarity  $\langle X, Y, R \rangle$  in Birkhoff's sense.

## 4 Urquhart's representation of lattices

In this section we briefly review the lattice representation of [12]. A *doubly ordered set* is a structure  $\langle X, \leq_1, \leq_2 \rangle$  such that  $\leq_1, \leq_2$  are quasiorders on  $X$  and

$$x \leq_1 y \text{ and } x \leq_2 y \text{ imply } x = y. \quad (4.1)$$

If the relations are clear from the context we shall name a doubly ordered set just by its base set.

If  $Y \subseteq X$  and  $i \in \{1, 2\}$ , we let  $\downarrow_i Y$  and  $\uparrow_i Y$  be the downset, respectively, the upset of  $Y$  with respect to  $\leq_i$ . Define two mappings  $l, r : 2^X \rightarrow 2^X$  by

$$l(Y) \stackrel{\text{df}}{=} \{x : \uparrow_1 x \cap Y = \emptyset\}, \quad (4.2)$$

$$r(Y) \stackrel{\text{df}}{=} \{x : \uparrow_2 x \cap Y = \emptyset\}. \quad (4.3)$$

$l$  and  $r$  can be viewed as intuitionistic negations; for example,  $l(Y)$  is the largest  $\leq_1$ -increasing subset of  $X$  disjoint from  $Y$ . These can be written in modal form as

$$l(Y) = [\leq_1](-Y), \quad (4.4)$$

$$r(Y) = [\leq_2](-Y). \quad (4.5)$$

$Y$  is called a *stable set*, if  $Y = l(r(Y))$ . The collection of stable sets is denoted by  $L_X$ . Observe that

$$l(r(Y)) = l([\leq_2](-Y)) = [\leq_1](-[\leq_2](-Y)) = [\leq_1]\langle \leq_2 \rangle(Y). \quad (4.6)$$

**Lemma 4.1.** [12] *Let  $(X, \leq_1, \leq_2)$  be a doubly ordered set.*

1. *The mappings  $l$  and  $r$  form a Galois connection between the lattice of  $\leq_1$ -increasing subsets of  $X$  and the lattice of  $\leq_2$ -increasing subsets of  $X$ .*
2. *If  $Y$  is  $\leq_2$  increasing, then  $l(Y)$  is a stable set.*

Thus, if  $Y$  is  $\leq_1$  increasing and  $Z$  is  $\leq_2$  increasing, then  $Y \subseteq l(Z)$  if and only if  $Z \subseteq r(Y)$ .

For  $Y, Z \in L_X$  let

$$Y \vee_X Z \stackrel{\text{df}}{=} [\leq_1]\langle \leq_2 \rangle(Y \cup Z), \quad (4.7)$$

$$Y \wedge_X Z \stackrel{\text{df}}{=} Y \cap Z. \quad (4.8)$$

**Theorem 4.2.** [12] *The structure  $\langle L_X, \vee_X, \wedge_X, \emptyset, X \rangle$  is a complete lattice.*

We call this structure the *Urquhart complex algebra of  $X$* , and denote it by  $\mathfrak{Cm}_U(X)$ .

Next, we go from lattices to frames. A *filter – ideal pair* is a pair  $\langle F, I \rangle$ , where  $F \in \mathcal{F}$ ,  $I \in \mathcal{I}$ , and  $F \cap I = \emptyset$ . Define the component-wise quasiorders on the set of all filter – ideal pairs by

$$\langle F_1, I_1 \rangle \preceq_1 \langle F_2, I_2 \rangle \stackrel{\text{df}}{\iff} F_1 \subseteq F_2, \quad (4.9)$$

$$\langle F_1, I_1 \rangle \preceq_2 \langle F_2, I_2 \rangle \stackrel{\text{df}}{\iff} I_1 \subseteq I_2, \quad (4.10)$$

and let  $\preceq$  be the intersection of  $\preceq_1$  and  $\preceq_2$ . A filter – ideal pair is called a *maximal pair*, if it is maximal with respect to  $\preceq$ .

**Lemma 4.3.** *Each filter – ideal pair can be extended to a maximal pair.*

The collection of all maximal pairs is denoted by  $X_L$ . To facilitate notation, if  $x = \langle F, I \rangle \in X_L$ , we let  $x_1 = F$  and  $x_2 = I$ . Furthermore, if  $x, y \in X_L$ , we let  $x \preceq_i y$  if and only if  $x_i \subseteq y_i$ . With these definitions  $\langle X_L, \preceq_1, \preceq_2 \rangle$  is a doubly ordered set, which we call the *Urquhart canonical frame of  $L$* , denoted by  $\mathfrak{Cf}_U(L)$ .

**Theorem 4.4.** [12] *Let  $L$  be a lattice and define  $h : L \rightarrow 2^{X_L}$  by  $h(a) \stackrel{\text{df}}{=} \{x \in X_L : a \in x_1\}$ . Then  $h$  is a lattice embedding into  $\mathfrak{Cm}_U \mathfrak{Cf}_U(L)$ .*

Whereas every lattice can be embedded into the complex algebra of its Urquhart canonical frame, an analogous embedding for doubly ordered frames is not always possible, even on the set level: Suppose that  $|X| \geq 2$ ,  $\leq_1$  is the identity, and  $\leq_2$  is the universal relation. Clearly,  $\langle X, \leq_1, \leq_2 \rangle$  is a doubly ordered frame. Since  $\leq_1$  is the identity,  $[\leq_1](Y) = Y$  for every  $Y \subseteq X$ . If  $Y \neq \emptyset$  and  $y \in Y$ , then  $x \leq_2 y$  for all  $x \in X$ , since  $\leq_2$  is the universal relation. Hence,  $\langle \leq_2 \rangle(Y) = X$  for every nonempty  $Y \subseteq X$ . It follows that  $L_X = \{X, \emptyset\}$ , and therefore,  $\mathfrak{Cf}_U \mathfrak{Cm}_U(X) = \{\langle \{X\}, \{\emptyset\} \rangle\}$ , which has only one element. Since  $|X| \geq 2$ , there is no injective mapping  $X \rightarrow \mathfrak{Cf}_U \mathfrak{Cm}_U(X)$ . Thus, we obtain the following theorem:

**Theorem 4.5.** *If  $|X| \geq 2$ , there is a doubly ordered frame  $X$  which cannot be embedded into  $\mathfrak{Cf}_U \mathfrak{Cm}_U(X)$ .*

## 5 Concept algebras and dicomplemented lattices

Formal concept analysis is based on the formalization of the notion of *concept*. Concepts are considered as basic unit of thought. They are determined by their extent and their intent. The extent consists of all objects belonging to the concept while the intent is the set of all attributes shared by all objects of the concept. Each of these sets (extents and intents) should uniquely define the corresponding concept. To develop a logic based on concepts as units of thought, the logical operators need to be formalized appropriately.

The universe of discourse is a binary relation involving objects and attributes of interest. The set of concepts of this context forms a complete lattice [13]. Towards developing a Boolean concept logic, the conjunction was encoded by the meet and the disjunction by the join in the concept lattice.  $\top$  corresponds to the top element and  $\perp$  to the bottom element of the concept lattice. To formalize negation we face the problem that the complement of extents are not always extents, and idem for intents. Two options for negation were considered:

- The first approach relaxes the definition of concept to accommodate the complement of extents/intents. This leads to the notion of semi-, pre-, protoconcepts and double Boolean algebras [14].
- The second approach obtains the negations as concepts generated by the complement of the extents or complement of the intents. This leads to concept algebras and dicomplemented lattices [14, 6].

Here we consider the second approach. If  $L = \mathfrak{Con}\langle X, Y, R \rangle$ , and  $\langle A, B \rangle \in L$ , we define mappings  $\triangleleft, \nabla : L \rightarrow L$  by

$$\langle A, B \rangle^{\triangleleft} = \langle f_R(-A), [[R^c]](-A) \rangle, \quad (5.1)$$

$$\langle A, B \rangle^{\nabla} = \langle [[R]](-B), g_R(-B) \rangle. \quad (5.2)$$

A *concept algebra* is a structure  $\langle L, +, \cdot, \triangleleft, \nabla, 0, 1 \rangle$  of type  $\langle 2, 2, 1, 1, 0, 0 \rangle$  which is isomorphic to a concept lattice with the additional operations defined by (5.1) and (5.2).

Generalizing these structures, Kwuida [6] has introduced the following class of algebras: A *dicomplemented lattice*<sup>1</sup> (DL) is a structure  $\langle L, +, \cdot, \triangleleft, \nabla, 0, 1 \rangle$  such that  $\triangleleft, \nabla : L \rightarrow L$  are operators such that for all  $a, b \in L$ ,

<sup>1</sup>These are called weakly dicomplemented lattices in [6]; here we follow the original notation [14].

$$a^{\Delta\Delta} \leq a, \quad (5.3) \quad a \leq a^{\nabla\nabla}, \quad (5.6)$$

$$a \leq b \Rightarrow b^{\Delta} \leq a^{\Delta}, \quad (5.4) \quad a \leq b \Rightarrow b^{\nabla} \leq a^{\nabla}, \quad (5.7)$$

$$a \leq (a \cdot b) + (a \cdot b^{\Delta}). \quad (5.5) \quad (a+b) \cdot (a+b^{\nabla}) \leq a. \quad (5.8)$$

**Theorem 5.1.** [14] *Each concept algebra is a dicomplemented lattice.*

**Lemma 5.2.** [3] *The following are equivalent:*

1. (5.6) and (5.7),
2.  $a \leq b^{\nabla} \iff b \leq a^{\nabla}$ .

Similarly,

**Lemma 5.3.** *The following are equivalent:*

1. (5.3) and (5.4),
2.  $a^{\Delta} \leq b \iff b^{\Delta} \leq a$ .

Every bounded lattice can be made into a DL by defining

$$a^{\Delta} = \begin{cases} 1, & \text{if } a \neq 1, \\ 0, & \text{if } a = 1, \end{cases} \quad (5.9)$$

$$a^{\nabla} = \begin{cases} 0, & \text{if } a \neq 0, \\ 1, & \text{if } a = 0. \end{cases} \quad (5.10)$$

These are called *trivial complementations*.

**Lemma 5.4.** [6, 14] *Let  $L$  be dicomplemented and  $a, b \in L$ . Then,  $^{\Delta\Delta}$  is an interior operator and  $^{\nabla\nabla}$  is a closure operator. Furthermore, for all  $a, b \in L$ ,*

$$a + a^{\Delta} = 1, \quad a \cdot a^{\nabla} = 0, \quad (5.11)$$

$$0^{\Delta} = 0^{\nabla} = 1, \quad 1^{\Delta} = 1^{\nabla} = 0, \quad (5.12)$$

$$(a \cdot b)^{\Delta} = a^{\Delta} + b^{\Delta}, \quad (a+b)^{\nabla} = a^{\nabla} \cdot b^{\nabla}, \quad (5.13)$$

$$a^{\Delta\Delta\Delta} = a^{\Delta}, \quad a^{\nabla\nabla\nabla} = a^{\nabla}, \quad (5.14)$$

$$a^{\Delta\nabla} \leq a^{\Delta\Delta} \leq a \leq a^{\nabla\nabla} \leq a^{\nabla\Delta}, \quad a^{\nabla} \leq a^{\Delta}, \quad (5.15)$$

$$a \cdot b^{\Delta} \leq b \Rightarrow a \leq b, \quad a \leq b + a^{\nabla} \Rightarrow a \leq b, \quad (5.16)$$

$$a \cdot b = 0 \Rightarrow a \leq b^{\Delta}, \quad a + b = 1 \Rightarrow a^{\nabla} \leq b, \quad (5.17)$$

$$a \cdot a^{\Delta} = 0 \Rightarrow a = a^{\Delta\Delta}, \quad a + a^{\nabla} = 1 \Rightarrow a = a^{\nabla\nabla}. \quad (5.18)$$



(5.12) and (5.13) show that  $\nabla$  is a sufficiency operator and  $\Delta$  is a dual sufficiency operator. Therefore, we can replace (5.4) and (5.7), by the equations in (5.12) and (5.13). In particular, the class of dicomplemented lattices is equational, and we denote it by  $\mathfrak{D}$ .

Let  $L$  be a DL. A proper filter  $F$  of  $L$  is called *primary* if  $a \in F$  or  $a^\Delta \in F$  for all  $a \in L$ . A proper ideal  $I$  is called *primary* if  $a \in I$  or  $a^\nabla \in I$  for all  $a \in L$ . The set of all primary filters, respectively, primary ideals, is denoted by  $\mathcal{F}_P$ , respectively,  $\mathcal{I}_P$ .

The following separation lemma and its corollary are decisive for the representation results based on Urquhart's representation:

**Lemma 5.5.** [6, Lemma 2.2.1] *If  $F \in \mathcal{F}$  and  $I \in \mathcal{I}$  with  $F \cap I = \emptyset$ , then there are a primary filter  $F'$  such that  $F \subseteq F'$ , and a primary ideal  $I'$  such that  $I \subseteq I'$  and  $F' \cap I' = \emptyset$ .*

*Proof.* Let  $F \in \mathcal{F}$  and  $I \in \mathcal{I}$  such that  $F \cap I = \emptyset$ . Let  $K = \{G \in \mathcal{F} : F \subseteq G \text{ and } G \cap I = \emptyset\}$ . Since  $F \in K$  and every chain in  $K$  has an upper bound in  $K$ , it contains a maximal element  $F'$ . Similarly, let  $I'$  be an ideal containing  $I$  and maximal with respect to  $I' \cap F = \emptyset$ ; then  $F' \cap I' = \emptyset$ . Assume that  $F'$  is not primary; then there is some  $a \in L$  such that  $a \notin F'$  and  $a^\Delta \notin F'$ . Since  $F'$  is maximal disjoint to  $I$ ,  $[F' \cup \{a\}]_{\text{filt}} \cap I \neq \emptyset$  and  $[F' \cup \{a^\Delta\}]_{\text{filt}} \cap I \neq \emptyset$ . Thus, there exist  $b, c \in F'$  such that  $b \cdot a, c \cdot a^\Delta \in I$ . Since  $F'$  is a filter,  $b \cdot c \in F'$ , and since  $I$  is an ideal we may suppose that  $b = c$ . Therefore,  $b \cdot a + b \cdot a^\Delta \in I$ . (5.5) now implies  $b \in I$ , contradicting that  $F' \cap I = \emptyset$ . Dually, it can be shown that  $I'$  is primary.  $\square$

**Corollary 5.6.** *If  $\langle F, I \rangle$  is a maximal pair (in Urquhart's sense), then both  $F$  and  $I$  are primary.*

*Proof.* Let  $\langle F, I \rangle$  be a maximal pair. By Lemma 5.5 there is a filter ideal pair  $\langle F', I' \rangle$  such that  $F \subseteq F' \in \mathcal{F}_P$  and  $I \subseteq I' \in \mathcal{I}_P$ . The maximality of  $\langle F, I \rangle$  implies that  $F = F'$  and  $I = I'$ .  $\square$

For later use, we introduce the following convention: If  $M \subseteq L$ , we let  $M^{\nabla^{-1}} = \{a : a^\nabla \in M\}$  and  $M^{\Delta^{-1}} = \{a : a^\Delta \in M\}$ .

**Lemma 5.7.**

1. *Let  $F$  be a proper filter. Then,  $F^{\nabla^{-1}}$  is a proper ideal of  $L$  disjoint from  $F$ .*
2. *Let  $I$  be a proper ideal. Then,  $I^{\Delta^{-1}}$  is a proper filter of  $L$  disjoint from  $I$ .*

*Proof.* 1. If  $1 \in F^{\nabla^{-1}}$ , then  $0 = 1^\nabla \in F$ , contradicting that  $F$  is proper. If  $a \in F^{\nabla^{-1}}$  and  $b \leq a$ , then  $a^\nabla \in F$  and  $a^\nabla \leq b^\nabla$ , since  $\nabla$  is antitone. If  $a, b \in F^{\nabla^{-1}}$ , then  $a^\nabla, b^\nabla \in F$ , and thus,  $a^\nabla \cdot b^\nabla \in F$  since  $F$  is a filter. By (5.13),  $a^\nabla \cdot b^\nabla = (a + b)^\nabla$ , showing that  $a + b \in F^{\nabla^{-1}}$ . If  $a \in F$  and  $a \in F^{\nabla^{-1}}$ , i.e.  $a^\nabla \in F$ , then  $a \cdot a^\nabla = 0 \in F$ , contradicting that  $F$  is proper.

2. This can be shown dually.  $\square$

## 6 A discrete representation for DLs

We are now going to establish the discrete representation theorem for DLs, and consider the reducts  $\langle L, \nabla \rangle$  and  $\langle L, \triangle \rangle$  separately. A  $\nabla$ -frame is a structure  $\langle X, \leq_1, \leq_2, C \rangle$  such that  $\langle X, \leq_1, \leq_2 \rangle$  is a doubly ordered set, and  $C$  is a binary relation on  $X$  satisfying the following conditions:

- FC1.  $(\forall x, y, z)[xCy \text{ and } y \leq_2 z \Rightarrow xCz]$ ,  $C ; \leq_2 \subseteq C$ ,  
 FC2.  $(\forall x, y)[xCy \Rightarrow (\exists z)(x \leq_1 z \text{ and } yCz)]$   $C \subseteq \leq_1 ; C^\sim$ .  
 FC3.  $(\forall x, y, z)[x \leq_2 y \text{ and } x \leq_2 z \Rightarrow zCy]$   $\geq_2 ; \leq_2 \subseteq C$ .

The class of all  $\nabla$ -frames is denoted by  $\text{Frm}_{\nabla}$ .

**Theorem 6.1.** *Let  $\mathcal{X} = \langle X, \leq_1, \leq_2, C \rangle$  be a  $\nabla$ -frame, and  $\langle L_X, \vee_X, \wedge_X, \emptyset, X \rangle$  be its Urquhart complex algebra. Furthermore, for  $Y \in L_X$  set*

$$Y^{\nabla x} \stackrel{\text{df}}{=} \{x \in X : (\forall y)[xCy \Rightarrow y \notin Y]\} = [C](-Y). \quad (6.1)$$

Then,  $Y^{\nabla x} \in L_X$  and  $\langle L_X, \vee_X, \wedge_X, \nabla^x, \emptyset, X \rangle$  satisfies (5.6) – (5.8).

*Proof.* The frame conditions and the definition of  $\nabla^x$  were presented in [3]; here we give a simpler proof.

$Y^{\nabla x} \in L_X$ : Since  $[\leq_1]\langle \leq_2 \rangle$  is a closure operator, we have  $Y^{\nabla x} \subseteq [\leq_1]\langle \leq_2 \rangle(Y^{\nabla x})$ , and only the reverse inclusion needs to be proved. Let  $x \in [\leq_1]\langle \leq_2 \rangle([C](-Y))$ , and assume that  $C(x) \not\subseteq -Y$ , i.e. that there is some  $y \in Y$  and  $xCy$ . By FC2, there is some  $s$  such that  $x \leq_1 s$  and  $yCs$ .

Now,  $x \in [\leq_1]\langle \leq_2 \rangle([C](-Y))$  implies that  $\uparrow_1 x \subseteq \langle \leq_2 \rangle[C](-Y)$ , and thus there is some  $t$  such that  $s \leq_2 t$  and  $C(t) \cap Y = \emptyset$ . From  $yCs$  and  $s \leq_2 t$  we obtain  $y(C ; \leq_2)t$ , and now FC1 implies that  $yCt$ . Using FC2 again, it follows that  $y(\leq_1 ; C^\sim)t$ , and thus, there is some  $z$  such that  $y \leq_1 z$  and  $tCz$ . Since  $Y \in L_X$  it is  $\leq_1$  closed, and thus,  $z \in Y$ . This contradicts  $C(t) \cap Y = \emptyset$ .

(5.6): First, note that

$$Y^{\nabla x \nabla x} = [C](-Y^{\nabla x}) = [C](-[C](-Y)) = [C]\langle C \rangle(Y) = \{x : (\forall y)[xCy \Rightarrow (\exists z)(z \in Y \text{ and } zCy)]\}.$$

Now, suppose that  $x \in Y$ , and  $xCy$ . By FC2, there is some  $z$  such  $x \leq_1 zC^\sim y$ . Since  $Y \in L_X$ , it is  $\uparrow_{\leq_1}$  increasing, and thus,  $x \in Y$  and  $x \leq_1 z$  imply  $z \in Y$ . The claim now follows from  $yCz$ .

(5.7): If  $Y \subseteq Z \subseteq X$ , then  $-Z \subseteq -Y$ . Since  $[C]$  is isotone, we obtain  $[C](-Z) \subseteq [C](-Y)$ .

(5.8): Let  $Y, Z \in L_X$ ; we need to show that  $(Y \vee_X Z) \cap (Y \vee_X Z^{\nabla x}) \subseteq Y$ . Set  $M = (Y \vee_X Z) \cap (Y \vee_X Z^{\nabla x}) = [\leq_1]\langle \leq_2 \rangle(Y \cup Z) \cap [\leq_1]\langle \leq_2 \rangle(Y \cup [C](-Z))$ . Now,

$$\begin{aligned} M &= [\leq_1]\langle \leq_2 \rangle(Y \cup Z) \cap [\leq_1]\langle \leq_2 \rangle(Y \cup [C](-Z)), \\ &= [\leq_1]\langle \leq_2 \rangle(Y \cup Z) \cap \langle \leq_2 \rangle(Y \cup [C](-Z)), && [\leq_1] \text{ distributes over } \cap, \\ &= [\leq_1][\langle \leq_2 \rangle(Y \cup \langle \leq_2 \rangle Z) \cap \langle \leq_2 \rangle(Y) \cup \langle \leq_2 \rangle[C](-Z))] && \langle \leq_2 \rangle \text{ distributes over } \cup, \\ &= [\leq_1][\langle \leq_2 \rangle Y \cup \langle \leq_2 \rangle Z \cap \langle \leq_2 \rangle[C](-Z)], && \cup \text{ distributes over } \cap \\ &= Y \cup \langle \leq_2 \rangle Z \cap \langle \leq_2 \rangle[C](-Z), && Y = [\leq_1]\langle \leq_2 \rangle(Y). \end{aligned}$$

We are done if we can show that  $\langle \leq_2 \rangle Z \cap \langle \leq_2 \rangle[C](-Z) = \emptyset$ , since  $Y = [\leq_1]\langle \leq_2 \rangle Y$ . Assume that  $x \in \langle \leq_2 \rangle Z$  and  $x \in \langle \leq_2 \rangle[C](-Z)$ . Then, there is some  $y \in Z$  such that  $x \leq_2 y$ ; also, there is some  $z$  such that  $x \leq_2 z$  and  $C(z) \cap Z = \emptyset$ . By FC3 we have  $zCy$ , contradicting  $C(z) \cap Z = \emptyset$ .  $\square$

If  $\mathcal{X}$  is a  $\nabla$ -frame, its complex algebra is the structure  $\langle L_X, \vee_X, \wedge_X, \nabla^x, \mathbf{0}, X \rangle$  which we denote by  $\mathfrak{Cm}_{\nabla}(X)$ .

Conversely, let  $L$  be a dicomplemented lattice and  $X_L$  the set of all maximal pairs. Define a relation  $C_L$  on  $X_L$  by  $x C_L y$  if and only if  $x_1^{\nabla^{-1}} \subseteq y_2$ .

**Theorem 6.2.** *Let  $L$  be a DL, and  $C_L$  the relation on its Urquhart canonical frame defined by  $x C_L y$  if and only if  $x_1^{\nabla^{-1}} \subseteq y_2$ . Then,  $C_L$  satisfies FC1 – FC3.*

*Proof.* It was shown in [3] that  $C_L$  satisfies FC1 – FC2. For FC3, suppose that  $x_2 \subseteq y_2$  and  $x_2 \subseteq z_2$ . We need to show that  $z C_L y$ , i.e.  $a^{\nabla} \in z_1$  implies  $a \in y_2$  for all  $a \in L$ ; thus, suppose that  $a^{\nabla} \in z_1$ . Then,  $a^{\nabla} \notin z_2$ , and therefore,  $a^{\nabla} \notin x_2$ , since  $x_2 \subseteq z_2$ . It follows that  $a \in x_2$ , since  $x_2$  is primary, and  $x_2 \subseteq y_2$  implies  $z C_L y$ .  $\square$

A  $\Delta$ -frame is a structure  $\langle X, \leq_1, \leq_2, Q \rangle$  such that  $\langle X, \leq_1, \leq_2 \rangle$  is a doubly ordered set, and  $Q$  is a binary relation on  $X$  satisfying the following conditions:

$$\begin{aligned} \text{FC1}' & \quad (\forall x, y, z)[x Q y \text{ and } y \leq_1 z \Rightarrow x Q z], & Q; \leq_1 \subseteq Q, \\ \text{FC2}' & \quad (\forall x, y, z)[z \leq_2 x \text{ and } x Q y \Rightarrow z Q y], & \leq_2; Q \subseteq Q, \\ \text{FC3}' & \quad \forall x, y)[x Q y \Rightarrow (\exists z)(x \leq_2 z \text{ and } y Q z)] & Q \subseteq \leq_2; Q^{\vee}. \\ \text{FC4}' & \quad (\forall x, y, z)[x \leq_1 y \text{ and } x \leq_1 z \Rightarrow y Q z \text{ and } z Q y] & \geq_1; \leq_1 \subseteq Q \cap Q^{\vee}. \end{aligned}$$

The class of all  $\Delta$ -frames is denoted by  $\text{Frm}_{\Delta}$ . The following lemma lists some properties of  $Q$  which we shall use later on.

**Lemma 6.3.** *Suppose  $Q$  satisfies FC1' – FC2'.*

1.  $Q = Q; \leq_1$ .
2.  $[Q](Y) = \uparrow_2 [Q](Y)$  for all  $Y \in L_X$ .

*Proof.* 1. Since  $\leq_1$  is reflexive,  $x Q y$  implies  $x \leq_1 x Q y$ . The other direction is just FC1'.

2. Since  $\leq_2$  is reflexive, we have  $[Q](Y) \subseteq \uparrow_2 [Q](Y)$ . For the other direction, let  $x \in \uparrow_2 [Q](Y)$ . Then, there is some  $y \in X$  such that  $y \leq_2 x$  and  $Q(y) \subseteq Y$ . Let  $x Q z$ ; then,  $y \leq_2 x Q z$ , and FC2' implies that  $y Q z$ . Therefore,  $z \in Y$  since  $Q(y) \subseteq Y$ .  $\square$

**Theorem 6.4.** *Let  $\mathcal{X} = \langle X, \leq_1, \leq_2, Q \rangle$  be a  $\Delta$ -frame, and  $\langle L_X, \vee_X, \wedge_X, \mathbf{0}, X \rangle$  be its Urquhart complex algebra. Furthermore, for  $Y \in L_X$  set*

$$Y^{\Delta x} \stackrel{\text{df}}{=} [\leq_1] \langle Q \rangle (-Y)^2. \quad (6.2)$$

*Then,  $Y^{\Delta x} \in L_X$  and  $\langle L_X, \vee_X, \wedge_X, \mathbf{0}, X \rangle$  satisfies (5.3) – (5.5).*

---

<sup>2 $\Delta x$</sup>  is the operator defined in [9, p. 269] for dual sufficiency.

*Proof.*  $Y^{\Delta x} \in L_X$ : We need to show that  $[\leq_1]\langle \leq_2 \rangle(Y^{\Delta x}) = Y^{\Delta x}$  for  $Y \in L_X$ . Since  $[\leq_1]\langle \leq_2 \rangle$  is a closure operator, we only consider the “ $\subseteq$ ” part. Let  $Y \in L_X$ . By Lemma 6.3(3),  $[Q](Y)$  is  $\leq_2$ -increasing, in other words,  $[\leq_2][Q](Y) = [Q](Y)$ . Thus,  $[\leq_1]\langle \leq_2 \rangle[Q](Y) = [\leq_1]\langle -[Q](Y) \rangle = [\leq_1]\langle Q \rangle(-Y) = Y^{\Delta x}$  is stable.

5.3: Let  $Y \in L_X$ ; then

$$Y^{\Delta x \Delta x} = [\leq_1]\langle Q \rangle(-Y^{\Delta x}) \quad (6.3)$$

$$= [\leq_1]\langle Q \rangle(-[\leq_1]\langle Q \rangle(-Y)) \quad (6.4)$$

$$= [\leq_1]\langle Q \rangle\langle \leq_1 \rangle[Q](Y) \quad (6.5)$$

$$= [\leq_1]\langle Q; \leq_1 \rangle[Q](Y) \quad \text{by (2.2)} \quad (6.6)$$

$$= [\leq_1]\langle Q \rangle[Q](Y) \quad \text{by Lemma 6.3(1)}. \quad (6.7)$$

Thus,

$$\langle Q \rangle[Q](Y) \subseteq \langle \leq_2 \rangle(Y) \Rightarrow Y^{\Delta x \Delta x} = [\leq_1]\langle Q \rangle[Q](Y) \subseteq [\leq_1]\langle \leq_2 \rangle(Y) = Y. \quad (6.8)$$

Thus, it suffices to show that  $\langle Q \rangle[Q](Y) \subseteq \langle \leq_2 \rangle(Y)$ . Let  $x \in \langle Q \rangle[Q](Y)$ ; then, there is some  $y \in X$  such that  $xQy$  and  $Q(y) \subseteq Y$ . By FC3', there is some  $z \in X$  such that  $x \leq_2 z$  and  $yQz$ . Now,  $Q(y) \subseteq Y$  implies that  $z \in Y$ , and it follows that  $\uparrow_2 x \cap Y \neq \emptyset$ , i.e.  $x \in \langle \leq_2 \rangle(Y)$ .

5.4: Suppose that  $Y \subseteq Z \subseteq X$ . Then,  $-Z \subseteq -Y$  and the fact that both  $[\leq_1]$  and  $\langle Q \rangle$  are isotone with respect to  $\subseteq$  implies that  $Z^{\Delta x} = [\leq_1]\langle Q \rangle(-Z) \subseteq [\leq_1]\langle Q \rangle(-Y) = Y^{\Delta x}$ .

5.5: We need to show that  $Y \subseteq (Y \wedge_X Z) \vee_X (Y \wedge_X Z^{\Delta x})$  for all  $Y, Z \in L_X$ . By definition of the lattice operations

$$\begin{aligned} (Y \wedge_X Z) \vee_X (Y \wedge_X Z^{\Delta x}) &= [\leq_1]\langle \leq_2 \rangle((Y \cap Z) \cup (Y \cap [\leq_1]\langle Q \rangle(-Z))), \\ &= [\leq_1]\langle \leq_2 \rangle(Y \cap (Z \cup [\leq_1]\langle Q \rangle(-Z))). \end{aligned}$$

Since  $Y = [\leq_1]\langle \leq_2 \rangle(Y)$  it is enough to show that  $X = Z \cup [\leq_1]\langle Q \rangle(-Z)$ . Thus, let  $x \in X$ , and assume that  $x \notin Z \cup [\leq_1]\langle Q \rangle(-Z)$ ; then,  $x \notin Z$  and  $x \notin [\leq_1]\langle Q \rangle(-Z)$ . Thus, there is some  $y$  such that  $x \leq_1 y$ , and  $y \notin \langle Q \rangle(-Z)$ , i.e.  $y \in [Q](Z)$ . By FC4' we have  $yQx$ , and thus,  $x \in Z$ , a contradiction.  $\square$

If  $L$  is a DL, define a relation  $Q_L$  on  $X_L$  by  $xQ_L y$  if and only if  $x_2^{\Delta-1} \subseteq y_1$ .

**Theorem 6.5.**  $Q_L$  satisfies FC1' – FC4'.

*Proof.* FC1': Let  $xQy$  and  $y \leq_1 z$ . Then,  $x_2^{\Delta-1} \subseteq y_1 \subseteq z_1$ , showing  $xQz$ .

FC2': Let  $x \leq_2 z$  and  $zQy$ . Then,  $x_2 \subseteq z_2$  and  $z_2^{\Delta-1} \subseteq y_1$ . Now,  $x_2 \subseteq z_2$  implies  $x_2^{\Delta-1} \subseteq z_2^{\Delta-1}$ , and thus,  $x_2^{\Delta-1} \subseteq y_1$ . It follows that  $xQy$ .

FC3': Let  $xQy$ , i.e.  $x_2^{\Delta-1} \subseteq y_1$ . First, assume that  $a \in y_2^{\Delta-1} \cap x_2$ . Then,  $a^\Delta \in y_2$  and  $a^{\Delta\Delta} \in x_2$ , since  $a^{\Delta\Delta} \leq a$  and  $x_2$  is an ideal. Hence,  $a^\Delta \in x_2^{\Delta-1} \subseteq y_1$ , contradicting  $a^\Delta \in y_2$ . It follows that  $\langle y_2^{\Delta-1}, x_2 \rangle$  is a filter – ideal pair which can be extended to a maximal pair  $z$ . Then,  $x_2 \subseteq z_2$  and  $y_2^{\Delta-1} \subseteq z_1$ , showing  $z \leq_2 zQ^*y$ .

FC4': Let  $x, y, z \in X_L$  such that  $x_1 \subseteq y_1$  and  $x_1 \subseteq z_1$ . Let  $a \in y_2^{\Delta-1}$ , i.e.  $a^\Delta \in y_2$ . Then,  $a^\Delta \notin y_1$ , and  $x_1 \subseteq y_1$  implies that  $a^\Delta \notin x_1$ . Since  $x_1$  is a primary filter by Corollary 5.6, it follows that  $a \in x_1$ . Hence,  $a \in z_1$ , which implies  $yQz$ . Reversing  $y$  and  $z$  shows that  $zQy$ .  $\square$

We can now state our representation result. A *DL-frame* is a structure  $\mathcal{X} = \langle X, \leq_1, \leq_2, C, Q \rangle$  such that  $\langle X, \leq_1, \leq_2 \rangle$  is a doubly ordered set, and  $C$  and  $Q$  are binary relations on  $X$  which satisfy, respectively, FC1 – FC3 and FC1' – FC4'. Its complex algebra  $\langle L_X, \vee_X, \wedge_X, \Delta^X, \nabla^X, \emptyset, X \rangle$  is denoted by  $\mathfrak{Cm}_{DL}(X)$ .

**Theorem 6.6.** *Each DL can be embedded into the complex algebra of its Urquhart canonical frame.*

*Proof.* Let  $h : L \rightarrow \mathfrak{Cm} \mathfrak{Cf}(L)$  be defined by  $h(a) = \{x \in X_L : a \in x_1\}$ . It was shown in [12] that  $h$  is a lattice embedding, so all that is left to show is that  $h$  preserves  $\Delta$  and  $\nabla$ . First, recall that for all  $a \in L$ ,

$$h(a)^{\Delta_{X_L}} = [\leq_1] \langle Q \rangle (X \setminus h(a)), \quad \text{Definition of } \Delta_{X_L} \quad (6.9)$$

$$= \{x : (\forall y)[x \leq_1 y \Rightarrow y \in \langle Q \rangle (X \setminus h(a))]\}, \quad \text{Definition of } [\leq_1] \quad (6.10)$$

$$= \{x : (\forall y)[x \leq_1 y \Rightarrow Q(y) \cap (X \setminus h(a)) \neq \emptyset]\}, \quad \text{Definition of } \langle Q \rangle \quad (6.11)$$

$$= \{x : (\forall y)[x \leq_1 y \Rightarrow (\exists z)(y_2^{-1} \subseteq z_1 \text{ and } a \notin z_1)]\}, \quad \text{Definition of } Q \quad (6.12)$$

$$h(a)^{\nabla_{X_L}} = [C](X \setminus h(a)), \quad \text{Definition of } \nabla_{X_L} \quad (6.13)$$

$$= \{x : (\forall y)[xCy \Rightarrow a \notin y_1]\}, \quad (6.14)$$

$$= \{x : (\forall y)[x_1^{\nabla^{-1}} \subseteq y_2 \Rightarrow a \notin y_1]\}. \quad (6.15)$$

$h(a^\Delta) \subseteq h(a)^{\Delta_{X_L}}$ : Let  $x \in h(a^\Delta)$ , i.e.  $a^\Delta \in x_1$ , and  $x_1 \subseteq y_1$ . Then,  $a^\Delta \in y_1$ , and thus,  $a^\Delta \notin y_2$ . It follows that  $a \notin y_2^{\Delta^{-1}}$ , and therefore,  $y_2^{\Delta^{-1}} \cap \downarrow a = \emptyset$ . By Lemma 4.3 there is a maximal pair  $z$  such that  $y_2^{\Delta^{-1}} \subseteq z_1$  and  $a \in z_2$ . This shows that  $yQz$  and  $z \notin h(a)$ .

$h(a)^{\Delta_{X_L}} \subseteq h(a^\Delta)$ : We prove the contraposition; thus, let  $x \notin h(a^\Delta)$ , i.e.  $a^\Delta \notin x_1$ . Then,  $x_1 \cap \downarrow a^\Delta = \emptyset$ , and by Lemma 4.3 there is some maximal pair  $y$  such that  $x_1 \subseteq y_1$  and  $a^\Delta \in y_2$ . By (6.12) it suffices to show that  $a \in z_1$  whenever  $y_2^{\Delta^{-1}} \subseteq z_1$ . Thus, let  $y_2^{\Delta^{-1}} \subseteq z_1$ . From  $a^\Delta \in y_2$  we obtain  $a \in y_2^{\Delta^{-1}}$ , and therefore,  $a \in z_1$ .

Next, we show that  $h$  preserves  $\nabla$  as well.

$h(a^\nabla) \subseteq h(a)^{\nabla_{X_L}}$ : Let  $x \in h(a^\nabla)$ , i.e.  $a^\nabla \in x_1$ . Suppose that  $xCy$ ; then  $x_1^{\nabla^{-1}} \subseteq y_2$  by definition of  $C$ . Now,  $a^\nabla \in x_1$  implies  $a \in x_1^{\nabla^{-1}}$ , and  $x_1^{\nabla^{-1}} \subseteq y_2$  implies  $a \in y_2$ . Since  $y_1 \cap y_2 = \emptyset$  it follows that  $a \notin y_1$ .

$h(a)^{\nabla_{X_L}} \subseteq h(a^\nabla)$ : Let  $x \in h(a)^{\nabla_{X_L}}$ , and assume that  $a^\nabla \notin x_1$ . Then,  $a \notin x_1^{\nabla^{-1}}$  which is a proper ideal by Lemma 5.7. Therefore, we can extend  $\langle \uparrow a, x_1^{\nabla^{-1}} \rangle$  to a maximal pair  $y$  by Lemma 4.3. Since  $x \in h(a)^{\nabla_{X_L}}$ , and  $x_1^{\nabla^{-1}} \subseteq y_2$ , we obtain  $a \notin y_1$  by (6.15), contradicting  $\uparrow a \subseteq y_1$ .  $\square$

## 7 Conclusion and outlook

We have shown that each dicomplemented lattice  $L$  can be embedded into the Urquhart complex algebra  $\mathfrak{Cm}_U \mathfrak{Cf}_U(L)$  of its canonical frame. Since its lattice reduct is a canonical extension of  $L$  [2] this also shows that the variety of dicomplemented lattices is canonical. In further work we intend to investigate the representation problem raised in [6]: Is every dicomplemented lattice embeddable into a concept algebra? We shall also endeavour to extend the logic based on doubly ordered frames presented in [9, Chapter 2.9] to the dicomplemented case.

In view of Theorem 4.5 it would be interesting to know whether there is an axiomatic extension of doubly ordered frames which would guarantee that  $\langle X, \leq_1, \leq_2 \rangle$  can be embedded into  $\mathfrak{Cf}_U \mathfrak{Cm}_U(X)$ . We suspect that the class of such frames is not first order axiomatizable.

## 8 References

- [1] Birkhoff, G.: *Lattice Theory*, vol. 25 of *Am. Math. Soc. Colloquium Publications*, 2 edition, AMS, Providence, 1948.
- [2] Craig, A., Haviar, M.: Reconciliation of approaches to the construction of canonical extensions of bounded lattices, *Mathematica Slovaca*, **6**, 2014, 1335–1356.
- [3] Dzik, W., Orłowska, E., van Alten, C.: Relational Representation Theorems for General Lattices with Negations, in: *Relations and Kleene Algebra in Computer Science* (R. A. Schmidt, Ed.), number 4136 in *Lecture Notes in Computer Science*, Springer Verlag, Berlin, 2006, ISBN 978-3-540-37873-0, 978-3-540-37874-7, 162–176.
- [4] Everett, C.: Closure Operators and Galois Theory in Lattices, *Transactions of the American Mathematical Society*, **55**(3), 1944, 514 – 525.
- [5] Gargov, G., Passy, S., Tinchev, T.: Modal Environment for Boolean Speculations, *Mathematical Logic and Applications* (D. Skordev, Ed.), Plenum Press, New York, 1987.
- [6] Kwuida, L.: *Dicomplemented Lattices. A Contextual Generalization of Boolean Algebras*, Ph. D. Thesis, Technische Universität Dresden, 2004.
- [7] Moore, E. H.: *Introduction to a form of general analysis*, Yale University Press, New Haven, 1910.
- [8] Ore, O.: Galois connexions, *Trans. Am. Math. Soc.*, **55**, 1944, 493–513.
- [9] Orłowska, E., Rewitzky, I., Radzikowska, A.: *Dualities for Structures of Applied Logics*, vol. 56 of *Studies in Logic*, College Publications, 2015.
- [10] Rasiowa, H.: *An Algebraic Approach to non-Classical Logics*, vol. 78 of *Studies in Logic and Foundations of Mathematics*, North Holland and PWN, 1974.
- [11] Rauszer, C.: Semi-Boolean algebras and their applications to intuitionistic logic with dual operations, *Fundamenta Mathematicae*, **83**, 1974, 219–249.
- [12] Urquhart, A.: A topological representation theorem for lattices, *Algebra Universalis*, **8**, 1978, 45–58.
- [13] Wille, R.: Restructuring lattice theory: An approach based on hierarchies of concepts, in: *Ordered sets* (I. Rival, Ed.), NATO Advanced Studies Institute, Reidel, Dordrecht, 1982, 445–470.
- [14] Wille, R.: Boolean Concept Logic, *Proc. 8th Int. Conf. on Conceptual Structures ICCS 2000* (B. Ganter, G. Mineau, Eds.), 1867, Springer-Verlag, Berlin, 2000.