Brock University

Department of Computer Science

Mixed Algebras and their Logics

Ivo Düntsch, Ewa Orloska, Tinko Tinchev

Brock University
Department of Computer Science
St. Catharines
Ontario Canada L2S 3A1

Institute of Telecommunications
Szachowa 1, 04-894
Warszawa, Poland

Faculty of Mathematics and Computer Science
Sofia University
Sofia, Bulgaria

Technical Report # CS-16-01
January 2016 (revised August 2016)

Mixed algebras and their logics

Ivo Düntsch*
Dept of Computer Science
Brock University
St Catharines, ON, L2S 3A1, Canada
duentsch@brocku.ca
Ewa Orłowska†
Institute of Telecommunications
Szachowa 1
04–894, Warszawa, Poland
orlowska@itl.waw.pl
Tinko Tinchev‡
Faculty of Mathematics and Computer Science
Sofia University
Sofia, Bulgaria
tinko@fmi.uni-sofia.bg

Dedicated to Luis Fariñas del Cerro

Abstract

We investigate complex algebras of the form $\langle 2^X, \langle R \rangle, [[S]] \rangle$ arising from a frame $\langle X, R, S \rangle$ where $S \subseteq R$, and exhibit their abstract algebraic and logical counterparts.

1 Introduction

Semantics of non-classical logics is provided either in terms of a class of algebras or a class of relational systems (frames. The theme of finding an equivalent (i.e. validating the same formulas) frame (resp. and algebraic) semantics once an algebraic (resp. frame) semantics is given has an extensive literature. One part of the problem – passing from algebraic to frame semantics – is a subject of correspondence theory [19]. Correspondence theory is well developed for logics whose algebraic semantics is based on distributive lattices, possibly with additional operators, and require first order definable relations in the corresponding

*Ivo Düntsch gratefully acknowledges support by the Natural Sciences and Engineering Research Council of Canada Discovery Grant 250153 and by the Bulgarian National Fund of Science, contract DID02/32/2009.
†Ewa Orłowska gratefully acknowledges partial support from the National Science Centre project DEC-2011/02/A/HS1/00395.
‡Tinko Tinchev gratefully acknowledges support by the Bulgarian National Fund of Science, contract DID02/32/2009.

frames such as standard modal logics, intuitionistic logic, some intermediate and relevant logics. Less is known for logics based on not necessarily distributive lattices. Equivalence of the two semantics can be obtained from a discrete duality between the two underlying classes of structures [12].

In this paper we discuss the problem of finding a frame semantics for logics whose algebraic semantics is based on what we call *PS-algebras*. These are Boolean algebras endowed with a normal and additive operator (a *possibility operator*) and a co-normal and co-additive operator (a *sufficiency operator*).

A special class of PS-algebras are the *mixed algebras* (MIAs). These were introduced in [3] and further investigated in [4]. The possibility part and the sufficiency part are related to each other by a second order property expressed in terms of their respective canonical extensions. We provide an equivalent characterization of mixed algebras in terms of the relations in their canonical frames.

Mixed algebras are not first order definable, and the complex algebras of their corresponding frames are not necessarily MIAs, so, MIAs and their frames treated as semantic structures of a formal language do not provide equivalent semantics for that language. However, for some axiomatic extensions of PS-algebras there are frames such that the equivalence holds. We discuss two of such classes, namely, the class of right ideal MIAs and the class **wMIA** of weak MIAs (wMIAs).We provide several universal-algebraic properties of those classes, in particular, we exhibit the equational class generated by **wMIA**.

It turns out that $\mathbf{Eq}(\mathbf{wMIA})$ provides an algebraic semantics for the logic $K^{\sim}$ which was developed based on the observation that the well known logic K as well as its sufficiency counterpart $K^{\star}$ presented in [16] are lacking in expressive power, and "necessity and sufficiency split the modal theory into two dual branches each of which spreads over less than a half of the Boolean realm" [5]. Finally, using the copying technique of [17] and the concept of special models of [5], we show that one frame relation suffices for wMIA frames: If $\langle B, f, g\rangle$ is a wMIA, then there is a frame $\langle X, R\rangle$ such that $\langle B, f, g\rangle$ and a subalgebra of $\langle 2^X, \langle R\rangle, [[R]]\rangle$ satisfy the same equations.

2 General definitions and notation

To make the paper more self–contained we recall a few concepts from Universal Algebra. Readers familiar with these concepts may skip straight to Section 3. Let $\mathfrak{F}$ be a signature of algebras, and X be a set of variables. The *set* $T_{\mathfrak{F}}(X)$ *of* $\mathfrak{F}$ *terms over* X is the smallest set such that

1. $X \subseteq T_{\mathfrak{F}}(X)$,
2. Each constant is in $T_{\mathfrak{F}}(X)$,
3. If $t_1, \ldots, t_n \in T_{\mathfrak{F}}(X)$ and $f \in \mathfrak{F}$ is n – ary, then $f(t_1, \ldots, t_n) \in T_{\mathfrak{F}}(X)$.

In the sequel we assume that $\mathfrak{F}$ is fixed, and we shall just write $T(X)$; we also assume that $T(X) \neq \emptyset$. Furthermore, $T(X)$ will be regarded as the absolutely free algebra over X with signature $\mathfrak{F}$, see [2, p.68].

If t is a term, we write $t(x_1, \ldots, x_n)$ if the variables occurring in t are among $x_1, \ldots, x_n$. Suppose that $\mathfrak{A}$ is an algebra of type $\mathfrak{F}$. If $t(x_1, \ldots, x_n) \in T(X)$, the *term function* $t^{\mathfrak{A}} : \mathfrak{A}^n \to \mathfrak{A}$ is defined as follows:

T_1. If t is the variable x_i, then $t^{\mathfrak{A}}(a_1, \ldots, a_n) = a_i$.

T_2. If $f \in \mathfrak{F}$ is k – ary and t has the form $f(t_1(x_1,\dots,x_n),\dots,t_k(x_1,\dots,x_n))$, then

$$t^{\mathfrak{A}}(a_1,\dots,a_n) = f^{\mathfrak{A}}(t_1^{\mathfrak{A}}(a_1,\dots,a_n),\dots,t_k^{\mathfrak{A}}(a_1,\dots,a_n)). \tag{2.1}$$

$t^{\mathfrak{A}}$ is called the *term function* of t (over $\mathfrak{A}$). For later use we mention the following fact:

Lemma 2.1. *[2, Theorem 10.3] Let $\mathfrak{A},\mathfrak{B}$ be algebras of the same type and $t(x_1,\dots,x_n)$ be an n – ary term.*

1. *Suppose that $a_i, b_i \in A$ for $1 \le i \le n$ and θ is a congruence on $\mathfrak{A}$. If $a_i \theta b_i$ for all $1 \le i \le n$, then*

$$t^{\mathfrak{A}}(a_1,\dots,a_n)\theta t^{\mathfrak{A}}(b_1,\dots,b_n).$$

2. *If $f : \mathfrak{A} \to \mathfrak{B}$ is a homomorphism, then $f(t^{\mathfrak{A}}(a_1,\dots,a_n)) = t^{\mathfrak{B}}(f(a_1),\dots,f(a_n))$.*

An *equation* (or *identity*, see [2, Definition 11.1]) is an expression of the form $\tau \approx \sigma$, where $\tau,\sigma \in T(X)$. If $\tau,\sigma \in T(X)$ are n – ary and $a_1,\dots,a_n \in A$, then the tuple $\langle a_1,\dots,a_n\rangle$ *satisfies the equation* $\tau \approx \sigma$ if $\tau^{\mathfrak{A}}(a_1,\dots,a_n) = \sigma^{\mathfrak{A}}(a_1,\dots,a_n)$. If $\tau^{\mathfrak{A}}(a_1,\dots,a_n) = \sigma^{\mathfrak{A}}(a_1,\dots,a_n)$ for all tuples $\langle a_1,\dots,a_n\rangle \in A^n$, we say that $\tau \approx \sigma$ *is true in* $\mathfrak{A}$, written as $\mathfrak{A} \models \tau \approx \sigma$.

As no generality is lost, we shall tacitly assume that a class of algebras is closed under isomorphic copies. If $\mathbf{K}$ is a class of algebras of the same type we denote by $\mathbf{H}(\mathbf{K})$ the collection of all homomorphic images of $\mathbf{K}$, by $\mathbf{S}(\mathbf{K})$ the collection of all subalgebras of $\mathbf{K}$, and by $\mathbf{P}(\mathbf{K})$ the collection of all products of elements of $\mathbf{K}$. The equational class $\mathbf{HSP}(\mathbf{K})$ generated by $\mathbf{K}$ is denoted by $\mathbf{Eq}(\mathbf{K})$. $Con(\mathfrak{A})$ is the set of all congruences on the algebra $\mathfrak{A}$.

Suppose that $\mathfrak{B} = \langle B,\wedge,\vee,\neg,0,1\rangle$ is a Boolean algebra. With some abuse of language we will usually identify algebras with their base set if no confusion can arise. Note that $a = b$ if and only if $\neg((a\wedge\neg b)\vee(b\wedge\neg a)) = 1$, and thus for each equation $\tau \approx \sigma$ there is an equation $\tau' \approx 1$ such that $B \models \tau \approx \sigma$ if and only if $B \models \tau' \approx 1$.

If $A \subseteq B$ and $f : B \to B$ is a function, then $f[A] = \{f(a) : a \in A\}$ is the *image of A under f*. The *dual of f* is the mapping $f^{\partial} : B \to B$ defined by $f^{\partial}(a) = \neg f(\neg a)$.

For the background of universal algebra we refer the reader to [2] and for frame and algebraic semantics of modal logics to [1] or [7].

3 Possibility and sufficiency algebras

In this section we review the concepts of possibility and sufficiency algebras and their canonical extensions.

Traditionally, a *modality* – or an *operator* [10] – on a Boolean algebra is a function $f : B \to B$ which satisfies $f(0) = 0$ (normal), and $f(a\vee b) = f(a)\vee f(b)$ (additive) for all $a,b \in B$. In recent years, however, many more operators with different properties have been considered in the study of modal logics, so that the term may mean almost any intensional operator on B. In this paper we shall be concerned with the two modalities *possibility* and *sufficiency* as well as operators definable from these and the Boolean operators.

A *possibility operator* on B is a normal and additive function $f : B \to B$; its dual f^{∂} is called a *necessity operator*. Clearly, a mapping $g : B \to B$ is a necessity operator if and only if $g(1) = 1$ and $g(a\wedge b) =$

$g(a) \wedge g(b)$ for all $a,b,\in B$. If f is a possibility operator on B, the pair $\langle B, f\rangle$ is called a *possibility algebra*. Dually, if u is a necessity operator on B, the pair $\langle B, u\rangle$ is called a *necessity algebra*.

A *sufficiency operator* on B is a function $g : B \to B$ which satisfies $g(0) = 1$ (co–normal), and $g(a \vee b) = g(a) \wedge g(b)$ (co–additive) for all $a, b \in B$. If g is a sufficiency operator on B, the pair $\langle B, g\rangle$ is called a *sufficiency algebra*. To the best of our knowledge, sufficiency operators were first introduced to modal logic by Humberstone [9]. In some sense, a sufficiency operator is the "complementary counterpart" to a possibility operator. This will be made clearer in the next section.

For a Boolean algebra B, we let $B^c = 2^{\mathrm{Ult}(B)}$ be its canonical extension [10], and $h : B \hookrightarrow B^c$ be the Stone embedding, i.e. $h(a) = \{F \in \mathrm{Ult}(B) : a \in F\}$. If $f, g : B \to B$ are operators on B, then two canonical extensions $f^\sigma, g^\pi : B^c \to B^c$ are defined by

$$f^\sigma(a) = \bigcup\{\bigcap\{h(f(x)) : x \in F\} : F \in a\}, \tag{3.1}$$

$$g^\pi(a) = \bigcap\{\bigcup\{h(g(x)) : x \in F\} : F \in a\}. \tag{3.2}$$

In particular, if $F \in \mathrm{Ult}(B)$, then

$$f^\sigma(\{F\}) = \bigcap\{h(f(x)) : x \in F\}, \tag{3.3}$$

$$g^\pi(\{F\}) = \bigcup\{h(g(x)) : x \in F\}. \tag{3.4}$$

There are representation theorems both for possibility and sufficiency algebras:

Theorem 3.1. *Suppose that B is a Boolean algebra.*

1. *[10] If f is a possibility operator on B, then, f^σ is a possibility operator on B^c, and the Stone mapping $h : \langle B, f\rangle \hookrightarrow \langle B^c, f^\sigma\rangle$ is an embedding.*
2. *[3] If g is a sufficiency operator on B, then, g^π is a sufficiency operator on B^c, and the Stone mapping $h : \langle B, g\rangle \hookrightarrow \langle B^c, g^\pi\rangle$ is an embedding.*

In particular, $h(f(a)) = f^\sigma(h(a))$ and $h(g(a)) = g^\pi(h(a))$ for all $a \in B$.

If f is a possibility operator on B and g a sufficiency operator, we call the structure $\langle B, f, g\rangle$ a *PS–algebra*, and $\langle B^c, f^\sigma, g^\pi\rangle$ its canonical extension. Theorem 3.1 tells us that $\langle B^c, f^\sigma, g^\pi\rangle$ is a PS–algebra, and h is an embedding of PS–algebras. For the rest of this section, we suppose that $\langle B, f, g\rangle$ is a PS–algebra.

If g is an operator on B we let $g^*(a) = g(\neg a)$. Note that g^* and g are mutually term definable. Furthermore, g is a sufficiency operator if and only if g^* is a necessity operator.

Theorem 3.2. *There is a 1 – 1 correspondence between PS – congruences on B and (Boolean) filters which are closed under f^∂ and g^*.*

Proof. It is well known (see e.g. [11]) that each Boolean congruence θ is uniquely determined by a filter F_θ, where

$$F_\theta = \{a \in B : a\theta 1\},$$

and, conversely, each filter F uniquely determines a congruence θ_F on B by

$$a\theta_F b \iff (\exists t)[t \in F \text{ and } a \wedge t = b \wedge t].$$

Furthermore, it was shown in [14] that a Boolean congruence θ preserves a necessity operator m if and only if F_θ is closed under m, i.e. $a \in F_\theta$ implies $m(a) \in F_\theta$. Clearly, θ preserves f if and only if it preserves f^∂, and θ preserves g if and only if it preserves g^*. Since both f^∂ and g^* are necessity operators, the claim follows. □

Define a mapping $u : B \to B$ by

$$u(a) = f^\partial(a) \wedge g^*(a) = f^\partial(a) \wedge g(\neg a). \tag{3.5}$$

Since both f^∂ and g^* are necessity operators, so is u.

A filter F of B is called a *u – filter*, if $a \in F$ implies $u(a) \in F$ for all $a \in B$. Theorem 3.2 now immediately implies

Corollary 3.3. *There is a 1 – 1 correspondence between congruences on B and u – filters.*

Proof. Let θ be a congruence on $\langle B, f, g\rangle$; then, F_θ is closed under f^∂ and g^* by Theorem 3.2. Thus, if $a \in F_\theta$, then $f^\partial(a), g^*(a) \in F_\theta$, and thus, $u(a) = f^\partial(a) \wedge g^*(a) \in F_\theta$, since F_θ is a filter.

Conversely, let F be a u – filter and $a \in F$. Then, $u(a) = f^\partial(a) \wedge g^*(a) \in F$ by the hypothesis, and thus, $f^\partial(a), g^*(a) \in F$ since F is a filter. Hence, θ_F is a PS – congruence, again by Theorem 3.2. □

4 Algebras and frames

The set of all binary relations on a set X is denoted by $\mathrm{Rel}(X)$; if $R_1, \ldots \in \mathrm{Rel}(X)$, the structure $\langle X, R_1, \ldots\rangle$ is called a *frame*. For $x \in X$, we let $R(x) = \{z \in X : xRz\}$. Relational composition and converse are denoted by ; , respectively by $\breve{}$; furthermore, $1'$ is the identity relation.

For $R \in \mathrm{Rel}(X)$, we define two operators on 2^X by

$$\langle R\rangle(S) = \{x : (\exists y)[xRy \text{ and } y \in S]\} = \{x : R(x) \cap S \neq \emptyset\}. \tag{4.1}$$
$$[[R]](S) = \{x : (\forall y)[y \in S \Rightarrow xRy]\} = \{x : S \subseteq R(x)\}. \tag{4.2}$$

We also set

$$[R](S) = \langle R\rangle^\partial(S) = \{x : R(x) \subseteq S\}. \tag{4.3}$$

It is well known that $\langle R\rangle$ is a complete possibility operator on the power set algebra of X [10], and that $[[R]]$ is a complete sufficiency operator [3]. Note that

$$[[R]](S) = [-R](X \setminus S), \tag{4.4}$$

so that it may be said that $[R]$ talks about the properties of R, while $[[R]]$ talks about the properties of $-R$ (see also the discussion in [9]).

The structure $\langle 2^X, \langle R\rangle\rangle$ is called the *full possibility (P) complex algebra of* $\langle X,R\rangle$, denoted by $\mathfrak{Cm}_P(X,R)$ or just by $\mathfrak{Cm}_P(X)$ if R is understood. Similarly, $\mathfrak{Cm}_S(X) = \langle 2^X, [[R]]\rangle$ is the *full sufficiency (S) complex algebra of* $\langle X,R\rangle$, and $\mathfrak{Cm}_{PS}(X) = \langle 2^X, \langle R\rangle, [[R]]\rangle$ is the *full PS - complex algebra of* $\langle X,R\rangle$. A *P* (*S*, *PS*) *complex algebra* is an algebra (isomorphic to) a subalgebra of some $\langle 2^X, \langle R\rangle\rangle$ ($\langle 2^X, [[R]]\rangle$, $\langle 2^X, \langle R\rangle, [[R]]\rangle$).

The question arises whether the canonical extension of a possibility or a sufficiency algebra is isomorphic to a structure $\langle 2^U, \langle R\rangle\rangle$ or $\langle 2^U, [[R]]\rangle$ for some frame $\langle U,R\rangle$. In both cases, the answer is positive, and the relation in question is uniquely determined:

Theorem 4.1. *1. [10, Theorem 3.10] If $\langle B, f\rangle$ is a possibility algebra, then there is, up to isomorphism, a unique relation R_f on* $\mathrm{Ult}(B)$ *such that $\langle R_f\rangle = f^\sigma$. This relation is defined by*

$$F R_f G \iff F \in f^\sigma(\{G\}). \tag{4.5}$$

The structure $\langle \mathrm{Ult}(B), R_f\rangle$ is called the P – canonical frame of $\langle B, f\rangle$.

2. *[3, Proposition 7] If $\langle B, g\rangle$ is a sufficiency algebra, then there is, up to isomorphism, a unique relation R_g on* $\mathrm{Ult}(B)$ *such that $[[R_g]] = g^\pi$. This relation is defined by*

$$F R_g G \iff F \in g^\pi(\{G\}). \tag{4.6}$$

The structure $\langle \mathrm{Ult}(B), R_g\rangle$ is called the S – canonical frame of $\langle B, g\rangle$.

The *PS – canonical frame* of a PS–algebra $\langle B, f, g\rangle$ is the structure $\langle \mathrm{Ult}(B), R_f, R_g\rangle$. Theorem 3.1 and Theorem 4.1 together now give us the following representation result:

This shows that the variety of PS–algebras is canonical in the sense of [8]. We also obtain the following representation theorem:

Theorem 4.2. *1. Each PS–frame is embeddable into the canonical frame of its complex algebra.*

2. *Each PS–algebra is embeddable into the complex algebra of its canonical frame.*

Here, a PS–frame is a structure $\langle U, R, S\rangle$ where R, S are binary relations on U, and its complex algebra is the power set algebra of U with additional operators $\langle R\rangle$ and $[[S]]$.

Finally in this section we mention an alternative description of the relations R_f and R_g of (4.5), respectively, (4.6) which does not explicitly use the canonical extension of B:

Lemma 4.3. *1. $\langle F, G\rangle \in R_f \iff f[G] \subseteq F$.*

2. *$\langle F, G\rangle \in R_g \iff F \cap g[G] \neq \emptyset$.*

Proof. 1. "⇒": This has been known for some time, see e.g. [20]. Suppose that $\langle F, G\rangle \in R_f$, i.e. $F \in f^\sigma(\{G\})$. Then, for all $x \in G$, $f(x) \in F$ by (3.3), which implies $f[G] \subseteq F$.

"⇐": Suppose $f[G] \subseteq F$; we need to show that $F \in \bigcap\{h(f(x)) : x \in G\}$. Let $x \in G$; then, $f(x) \in F$ by our hypothesis, and thus, $F \in h(f(x))$.

2. "⇒": Let $\langle F, G\rangle \in R_g$, i.e. $F \in g^\pi(\{G\})$. By (3.4), there is some $x \in G$ such that $g(x) \in F$, in other words, $F \cap g[G] \neq \emptyset$.

"⇐": Let $F \cap g[G] \neq \emptyset$, say, $x \in G$ and $g(x) \in F$. Then, $F \in h(g(x)) \subseteq \bigcup\{h(f(y)) : y \in G\} = g^\pi(\{G\})$. $\square$

5 The class MIA

Suppose that $\langle B, f, g\rangle$ is a PS–algebra. In the general definition, there is no relation between f and g, and between their associated canonical frames $\langle \mathrm{Ult}(B), R_f\rangle$ and $\langle \mathrm{Ult}(B), R_g\rangle$. Of course, such connections may exist: Consider, for example, the condition

$$f(a) = \neg g(a). \tag{5.1}$$

It is not hard to see that the corresponding frame $\langle \mathrm{Ult}(B), R_f, R_g\rangle$ satisfies the condition

$$R_f = \mathrm{Ult}(B)^2 \setminus R_g, \tag{5.2}$$

and that the respective representations for algebras satisfying (5.1) and frames satisfying (5.2) hold (see also Proposition 8 of [3]).

While the possibility algebras are the algebraic models of the logic K and the sufficiency algebras are the algebraic models of its sufficiency counterpart $K^\star$ [16], both are limited in their powers of expression if considered separately. For example, $\langle 2^X, \langle R\rangle\rangle$ can express reflexivity by

$$R \text{ is reflexive } \iff Y \subseteq \langle R\rangle(Y),$$

but it cannot express irreflexivity of R. On the other hand, $\langle 2^X, [[R]]\rangle$ can express irreflexivity by

$$R \text{ is irreflexive } \iff [[R]](Y) \subseteq -Y,$$

but not reflexivity. Neither $\langle 2^X, \langle R\rangle\rangle$ nor $\langle 2^X, [[R]]\rangle$ can express antisymmetry on its own, but together they can [4]:

$$R \text{ is antisymmetric } \iff \langle R\rangle([[R]](-Y) \cap Y) \subseteq Y.$$

Thus, it is worthwhile to consider the PS–algebras $\langle 2^X, \langle R\rangle, [[R]]\rangle$ obtained from a frame $\langle X, R\rangle$ with a single distinguished relation. Let us denote the class of complex algebras of this form by **CMIA**.

Next, let us step back and consider a PS–algebra $\mathfrak{B} = \langle B, f, g\rangle$ as a starting point. In [3], $\mathfrak{B}$ was called a *mixed algebra* (MIA), if in its PS – canonical frame $\langle \mathrm{Ult}(B), R_f, R_g\rangle$, the relations R_f and R_g were equal, and therefore, the full complex algebra of its canonical frame was of the form $\langle 2^{\mathrm{Ult}(B)}, \langle R\rangle, [[R]]\rangle$, where $R = R_f = R_g$; in other words, it is in **CMIA**. The following result now follows immediately from Theorem 4.1:

Theorem 5.1. *[3, 10] Let $\langle B, f, g\rangle$ be a PS–algebra. Then, there is a relation R on $\mathrm{Ult}(B)$ such that $\langle R\rangle = f^\sigma$ and $[[R]] = g^\pi$ if and only if $f^\sigma(\{G\}) = g^\pi(\{G\})$ for all $G \in \mathrm{Ult}(B)$. Furthermore, the relation R is unique with this property.*

The class of mixed algebras is denoted by **MIA**. Note that the MIA condition $R_f = R_g$ is a second order axiom. Indeed, it was shown in [4] that **MIA** is not first order axiomatizable. Observe that $\mathfrak{B}$ is a MIA if and only if for all $F, G \in \mathrm{Ult}(B)$,

$$f[G] \subseteq F \iff F \cap g[G] \neq \emptyset \tag{5.3}$$

by Lemma 4.3.

Starting with a MIA leads to a canonical frame $\langle \mathrm{Ult}(B), R_f, R_g\rangle$ with $R_f = R_g$. On the other hand, using a frame $\langle X, R\rangle$ as a starting point and considering the complex algebra $\langle 2^X, \langle R\rangle, [[R]]\rangle$ will not necessarily lead to a MIA since not every algebra in **CMIA** is in **MIA**, as the following example shows:

Example 1. This is based on Proposition 14 of [3]: Let X be infinite, and $R = 1'$. If $\mathfrak{Cm}_{PS}(X)$ is a MIA, then, by (4.5) and (4.6), we must have $R_{\langle R \rangle} = R_{[[R]]}$.

Suppose that F, G are ultrafilters of 2^X. Since R is the identity relation on X, $\langle R \rangle(a) = a$ for all $a \subseteq X$, hence, $\langle R \rangle[G] \subseteq F$ if and only if $F = G$. Suppose that $a \in G$, $|a| > 1$. Then, $x \in [[R]](a) \iff a \subseteq R(x) = \{x\}$, and it follows that $[[R]](a) = \emptyset$. Thus, if G is non principal, then $G \cap [[R]][G] = \emptyset$ and it follows that $\langle 2^X, \langle R \rangle, [[R]] \rangle$ does not satisfy (5.3).

Similarly, if $R = (X \times X) \setminus 1'$, then $\langle R \rangle(a) = X$ for all a with $|a| > 1$, and thus, $\langle R \rangle[G] \subseteq F$ for all non–principal $G \in \mathrm{Ult}(2^X)$ and all $F \in \mathrm{Ult}(2^X)$; in particular, $\langle R \rangle[G] \subseteq G$. On the other hand, $[[R]](a) = X \setminus a$ for all $a \subseteq X$, so that $G \cap [[R]][G] = \emptyset$. $\square$

Thus, not every PS – complex algebra of a structure $\langle X, R \rangle$ is a MIA, and we cannot have a general discrete duality theorem between PS -frames $\langle X, R, R \rangle$ and canonical frames of complex algebras of $\langle 2^X, \langle R \rangle, [[R]] \rangle$.

It is unknown which class of frames $\langle X, R \rangle$ have a full PS–complex algebra in **MIA**. A general characterization needs to be second order, since **MIA** is not first order axiomatizable. The only general property we know which leads to a MIA is that of right ideal frames. Set $\mathbf{1} = X \times X$. A binary relation R on X is called a *right ideal relation*, if $R\ ;\ \mathbf{1} \subseteq R$, and the pair $\langle X, R \rangle$ is called a *right ideal frame*. The following observation is already (implicitly) contained in [15], p. 79:

Lemma 5.2. *R is a right ideal relation if and only if $\langle R \rangle(X) = [[R]](X)$.*

Proof. "$\Rightarrow$": Let $x \in \langle R \rangle(X)$; then, $R(x) \neq \emptyset$. If, say, xRy and $z \in X$, then $xRy\mathbf{1}z$, and $R\ ;\ \mathbf{1} \subseteq R$ implies that xRz. Hence, $X \subseteq R(x)$, and thus, $x \in [[R]](X)$. The other direction follows from Lemma 6.1 below.

"$\Leftarrow$": Suppose that xRy and $z \in X$; we need to show that xRz. Since xRy, we have $R(x) \neq \emptyset$, hence, $x \in \langle R \rangle(X)$. The hypothesis implies that $x \in [[R]](X)$, hence, $X \subseteq R(x)$; in particular, xRz. $\square$

A PS–algebra $\langle B, f, g \rangle$ is called a *right ideal algebra* if $f(1) = g(1)$.

Lemma 5.3. *A right ideal algebra $\langle B, f, g \rangle$ is a MIA.*

Proof. We have to show the "$\Rightarrow$" direction of (5.3): Suppose that F, G are ultrafilters of B, and that $f[G] \subseteq F$. Then, in particular, $f(1) \in F$, and thus, $g(1) \in F$ since B is a right ideal algebra. Now, $1 \in G$ implies that $F \cap g[G] \neq \emptyset$. $\square$

Since the complex algebra of a right ideal frame is a right ideal algebra by Lemma 5.2, we immediately obtain

Theorem 5.4. *The PS – complex algebra of a right ideal frame $\langle X, R \rangle$ is a right ideal algebra.*

Theorem 5.5. *The PS – canonical frame of a right ideal algebra $\langle B, f, g \rangle$ is a right ideal frame.*

Proof. Let $X = \mathrm{Ult}(B)$. In view of Lemma 6.1 below it suffices to show that $\langle R \rangle(X) \subseteq [[R]](X)$. Let $F \in \langle R \rangle(X)$; then, there is some $G \in X$ such that FRG. Since B is a MIA, $R = R_f$, and thus, $f[G] \subseteq F$, in particular, $f(1) \in F$. Since B is a right ideal algebra it follows that $g(1) \in F$ as well. We need to show that $F \in [[R]](X)$, in other words that $X \subseteq R(F)$. Let $H \in X$; then, $1 \in H$ and $g(1) \in F$ shows that $F \cap g[H] \neq \emptyset$, hence, FR_gH. $\square$

6 The class wMIA

As the class **MIA** is too narrow to fully describe the properties of the class **CMIA**, let us start with the properties of $\langle 2^X, \langle R\rangle, [[R]]\rangle \in$ **CMIA**. The following observation shows how these algebras differ from MIAs:

Lemma 6.1. *1. For all $x \in X$,*

$$\langle R\rangle(\{x\}) = [[R]](\{x\}). \tag{6.1}$$

2. Let $A, B \subseteq X$ such that $A \cap B \neq \emptyset$. Then, $[[R]](A) \subseteq \langle R\rangle(B)$.

Proof. 1. "$\subseteq$": let $y \in \langle R\rangle(\{x\})$, i.e. yRx. Then, $\{x\} \subseteq R(y)$, which shows that $y \in [[R]](\{x\})$.

"$\supseteq$": $y \in [[R]](\{x\})$. Then, $\{x\} \subseteq R(y)$, and thus yRx. It follows that $y \in \langle R\rangle(\{x\})$.

2. Let $x \in A \cap B$; then, $\{x\} \subseteq A \cap B$. Since $[[R]]$ is a sufficiency operator, we have $[[R]](A) \subseteq [[R]](\{x\}$, and the fact that $\langle R\rangle$ is a possibility operator implies $\langle R\rangle(\{x\} \subseteq \langle R\rangle(B)$. The conclusion now follows from (6.1). □

Note that $\langle R\rangle(\{x\}) = [[R]](\{x\})$ only implies that in the canonical extension of $\mathfrak{Cm}_{PS}(X)$ we obtain that $\langle R\rangle^{\sigma}(F) = [[R]]^{\pi}(F)$ only for principal ultrafilters F of $\mathfrak{Cm}_{PS}(X)$. Example 1 shows that it need not hold for non–principal ultrafilters.

These observations lead to the following definition: A *weak mixed algebra* (wMIA) is a PS–algebra $\langle B, f, g\rangle$ such that

$$(\forall a, b)[a \wedge b \neq 0 \Rightarrow g(a) \leq f(b)]. \tag{6.2}$$

We shall denote the class of weak MIAs by **wMIA**. Note that, unlike **MIA**, the class **wMIA** is first order axiomatizable, indeed, it is a universal class. There are several characterizations of weak MIAs:

Theorem 6.2. *Let $\langle B, f, g\rangle$ be a PS–algebra. The following are equivalent:*

1. *B is a weak MIA.*
2. *$R_g \subseteq R_f$.*
3. *$g^{\pi}(\{F\}) \subseteq f^{\sigma}(\{F\})$ for all $F \in \mathrm{Ult}(B)$.*
4. *$(\forall a \in B)[a \neq 0 \Rightarrow g(a) \leq f(a)]$.*

Proof. 1. $\Rightarrow$ 2.: Let $F \cap g[G] \neq \emptyset$ and $a \in G$ with $g(a) \in F$. Suppose that $b \in G$; since $a \in G$ as well, we have $a \wedge b \neq 0$. It follows from (6.2) that $g(a) \leq f(b)$, and $g(a) \in F$ now implies that $f(b) \in F$.

2. $\Rightarrow$ 3.: This follows immediately from the definitions of R_f and R_g in (4.5) and (4.6).

3. $\Rightarrow$ 1: Suppose that $a \wedge b \neq 0$, and assume that $g(a) \not\leq f(b)$, i.e. $g(a) \wedge \neg f(b) \neq 0$. Then, there are ultrafilters F, G such that $g(a), \neg f(b) \in F$ and $a, b \in G$. Then, $F \cap g[G] \neq \emptyset$, and thus, it follows from Lemma 4.3(2) and the definition of R_g that $F \in g^{\pi}(\{G\})$. Then, by the hypothesis, $F \in f^{\sigma}(\{G\})$, and it

follows from the definition of R_f and Lemma 4.3(1) that $f[G] \subseteq F$. Since $b \in G$ it follows that $f(b) \in F$, contradicting that $\neg f(b) \in F$.

Finally, we show that 4. $\Rightarrow$ 1., the other direction being trivial: Suppose that 4. holds, and that $a \wedge b \neq 0$. Then, since g is antitone and f is isotone,

$$g(a) \leq g(a \wedge b) \stackrel{4.}{\leq} f(a \wedge b) \leq f(b).$$

This completes the proof. □

Observe that Theorem 6.2(3) shows that every MIA is a weak MIA. Since $\mathfrak{Cm}_{PS}(\langle X, R \rangle)$ is a weak MIA, Theorem 6.2(2) shows that for all ultrafilters F, G of 2^X in a weak MIA

$$F \cap [[R]][G] \neq \emptyset \Rightarrow \langle R \rangle [G] \subseteq F. \tag{6.3}$$

Theorem 6.2(2) suggests that we call a PS–frame $\langle X, R, S \rangle$ a *weak MIA frame*, if $S \subseteq R$. Even though we use two relations, we have a connection between R and S by $S \subseteq R$ which is one direction of the MIA condition. Our next result shows the correspondence between weak MIA frames and weak MIAs:

Lemma 6.3. *1. The complex algebra of a weak MIA frame is a weak MIA.*

2. The canonical frame of a weak MIA is a weak MIA frame.

Proof. 1. Suppose that $\langle X, R, S \rangle$ is a weak MIA frame. Let $\emptyset \neq Y \subseteq X$ and $x \in [[S]](Y)$. By 6.2(4) it is sufficient to show $x \in \langle R \rangle (Y)$. Since $x \in [[S]](Y)$, we obtain $Y \subseteq S(x)$, and therefore, $Y \subseteq R(x)$ by the hypothesis. It now follows from $Y \neq \emptyset$ that $R(x) \cap Y \neq \emptyset$, hence, $x \in \langle R \rangle (Y)$.

2. Suppose that $\langle B, f, g \rangle$ is a weak MIA. By Theorem 6.2(3) $\langle \mathrm{Ult}(B), R_f, R_g \rangle$ is a weak MIA frame. □

This gives us the representation theorem:

Theorem 6.4. *1. Each weak MIA frame is embeddable into the canonical frame of its complex algebra.*

2. Each weak MIA is embeddable into the complex algebra of its canonical frame.

wMIA is closed under subalgebras and homomorphic images, but not under products, as we shall see below.

7 The equational class generated by wMIA

In this section we shall describe the equational class generated by **wMIA**. First, we show that a weak MIA is a discriminator algebra. Recall the mapping $u : B \to B$ defined in (3.5), namely,

$$u(a) = f^{\partial}(a) \wedge g(\neg a). \tag{7.1}$$

It will turn out that u^{∂} is the unary discriminator. We have chosen to start with u as this mapping will be important later.

Theorem 7.1. *Let $\langle B, f, g\rangle$ be a PS–algebra. Then, B is a weak MIA if and only if*

$$u(a) = \begin{cases} 1, & \textit{if } a = 1, \\ 0, & \textit{otherwise}. \end{cases} \tag{7.2}$$

Proof. "$\Rightarrow$": First, consider $a = 1$. Then,

$$u(1) = f^{\partial}(1) \wedge g(0) = \neg f(0) \wedge g(0) = 1,$$

since $f(0) = 0$, and $g(0) = 1$. Next, let $a \neq 1$. Then, $\neg a \neq 0$, and

$$\begin{aligned} g(\neg a) &\leq f(\neg a) && \text{By Theorem 6.2} \\ \neg f(\neg a) \wedge g(\neg a) &= 0 \\ f^{\partial}(a) \wedge g(\neg a) &= 0 \\ u(a) &= 0, \end{aligned}$$

"$\Leftarrow$": Suppose that $a \neq 0$. By Theorem 6.2 it suffices to show that $g(a) \leq f(a)$. From $a \neq 0$ it follows that $\neg a \neq 1$, and thus, $u(\neg a) = 0$ by the hypothesis. Now, by the definition of u,

$$u(\neg a) = 0 \Longleftrightarrow f^{\partial}(\neg a) \wedge g(a) = 0 \Longleftrightarrow \neg f(a) \wedge g(a) = 0 \Longleftrightarrow g(a) \leq f(a).$$

This completes the proof. □

Theorem 7.1 gives us yet another characterization of weak MIAs among PS–algebras.

Corollary 7.2. *Each weak MIA is a discriminator algebra.*

Proof. Let B be a weak MIA. We show that B has a unary discriminator, i.e. there is a mapping $t : B \to B$ for which

$$t(a) = \begin{cases} 0, & \text{if } a = 0, \\ 1, & \text{otherwise}. \end{cases}$$

Indeed, set $t(a) = u^{\partial}(a) = \neg u(\neg a)$. Then, t fulfills the condition. □

Observe that it follows that **wMIA** is not an equational class, since every discriminator algebra is simple. To describe $\mathbf{Eq}(\mathbf{wMIA})$ we shall relax the condition that u^{∂} is the unary discriminator to the fact that u is an S5 necessity operator. Call a PS–algebra $\langle B, f, g\rangle$ a *$\tilde{K}$–algebra* if u satisfies the following conditions:

$$u(a) \leq a, \tag{7.3}$$

$$u(a) \leq u(u(a)), \tag{7.4}$$

$$a \leq u(u^{\partial}(a)). \tag{7.5}$$

The class of $K^{\sim}$–algebras is denoted by **KMIA**. The motivation for these algebras comes from the axiom system of the logic $K^{\sim}$ of [5] which we shall discuss below.

It follows immediately from Theorem 7.1 that a weak MIA satisfies (7.3) – (7.5). Since **KMIA** is an equational class and **wMIA** is not, the inclusion **wMIA** $\subseteq$ **KMIA** is strict. It may be instructive to present a concrete example:

Example 2. Suppose that $|B| > 2$, and let f be the identity on B and g be the Boolean complement. Then, f is a possibility operator, g is a sufficiency operator, and therefore, $\langle B, f, g\rangle$ is a PS–algebra. Furthermore, $f = f^{\partial}$, and, for all $a \in B$,

$$u(a) = f^{\partial}(a) \wedge g(-a) = a \wedge g(-a) = a, \tag{7.6}$$

and thus, $u^{\partial} = u$. Clearly, u satisfies (7.3) – (7.5), but is not a weak MIA. $\square$

The next result exhibits the precise connection between **wMIA** and **KMIA**:

Theorem 7.3. ***Eq(*wMIA*)* = KMIA.**

Proof. We shall show that

1. **KMIA** is semisimple, i.e. every subdirectly irreducible $K^{\sim}$ algebra is simple, and
2. The simple elements of **KMIA** are in **wMIA**.

Then, by Birkhoff's Theorem (see e.g. [2, Theorem 11.12]), every $K^{\sim}$ algebra is isomorphic to a subdirect product of weak MIAs, and thus, it is in the equational class generated by **wMIA**. The other direction follows from **wMIA** $\subseteq$ **KMIA**.

Let $\langle B, f, g\rangle \in$ **KMIA** be subdirectly irreducible. By Corollary 3.3, the congruences of B are in 1 – 1 correspondence with the u – filters of B, and therefore, $\langle B, u\rangle$ is subdirectly irreducible in the class of all Boolean algebras with an additional necessity operator. By (7.3) and (7.4) we have $u(a) = a \wedge u(a) \wedge u(u(a)) \wedge \ldots \wedge u^n(a)$, and therefore,

$$(\exists c \neq 1)(\forall a \neq 1) u(a) \leq c \tag{7.7}$$

by Rautenberg's criterion [13, p. 155]. By (7.4) we may suppose that $u(c) = c$. Assume that $c \neq 0$. Then, $\neg c \neq 1$, and

$$\neg c \underset{(7.5)}{\leq} u(u^{\partial}(\neg c)) = u(\neg u(c)) \underset{u(c)=c}{=} u(\neg c) \underset{(7.7)}{\leq} c, \tag{7.8}$$

a contradiction. It follows that $u(a) = 0$ for all $a \neq 1$, and, clearly, $u(1) = 1$. Hence, B is in **wMIA** by Theorem 7.1. $\square$

We close this section by showing that **KMIA** is closed under canonical extensions by describing the canonical frames. Call a frame $\langle X, R, S\rangle$ a **KMIA** *frame* if $R \cup -S$ is an equivalence relation.

Theorem 7.4. *1. Let $\langle B, f, g\rangle$ be in* **KMIA***, and $\langle \mathrm{Ult}(B), R_f, R_g\rangle$ be its canonical frame. Then, $R_f \cup -R_g$ is an equivalence relation.*

2. *Let* $\langle X, R, S\rangle$ *be a* **KMIA** *frame. Then,* $\langle 2^X, \langle R\rangle, [[S]]\rangle$ *is in* **KMIA**.

Proof. 1. Let w be the dual of u; then, by the properties of u, w is normal additive closure operator in which every open set is closed. It is well known from the properties of S5 that the canonical relation R_w on $\mathrm{Ult}(B)$ is an equivalence relation. Note that $\langle F, G\rangle \in R_w$ if and only if $w[G] \subseteq F$. We are going to show that $R_w = R_f \cup -R_g$:

"$\subseteq$": Assume that this is not true, i.e. that there are $F, G \in \mathrm{Ult}(B)$ such that

1. $\langle F, G\rangle \in R_w$, i.e. $(\forall a)[a \in G$ implies $f(a) \vee \neg g(a) \in F]$.
2. $\langle F, G\rangle \notin R_f$, i.e. $(\exists b)[b \in G$ and $f(b) \notin F]$.
3. $\langle F, G\rangle \in R_g$, i.e. $(\exists c)[c \in G$ and $g(c) \in F]$.

Let $d = b \wedge c$; then, $d \leq b$. Since $d \in G$ we have $f(d) \vee \neg g(d) \in F$. If $f(d) \in F$, then $d \leq b$ implies $f(b) \in F$, contradicting 2. If, on the other hand, $\neg g(d) \in F$, then $d \leq c$ implies $g(c) \leq g(d)$, since g is antitone. It follows from 3. that $g(d) \in F$, also a contradiction.

"$\supseteq$": Let $\langle F, G\rangle \in R_f$; then $f[G] \subseteq F$. Suppose that $a \in G$. Then, $f(a) \in F$ implies $f(a) \vee \neg g(a) \in F$, since F is a filter. It follows that $\langle F, G\rangle \in R_w$, and consequently, $R_f \subseteq R_w$. Next, assume that $-R_g \not\subseteq R_w$. Then, there is some pair $\langle F, G\rangle$ such that $F \cap g[G] = \emptyset$ and $w[G] \not\subseteq F$. The latter implies that there is some $a \in G$ with $w(g) \notin F$. By definition of w this implies in particular that $\neg g(a) \notin F$, thus, $g(a) \in F$, since F is prime. Together with $a \in G$ this contradicts $F \cap g[G] = \emptyset$.

2. Let $\langle X, R, S\rangle$ be a **KMIA** frame, and define the mapping $[U]] : 2^X \to 2^X$ by $[U]](Y) = [R](Y) \cap [[S]](-Y)$. We need to show that $[U]]$ satisfies (7.3) – (7.5):

(7.3): Let $x \in [U]](Y)$. Then, $x \in [R](Y)$ and $x \in [[S]](-Y)$. By definition of $[R]$ and $[[S]]$, this implies $R(x) \subseteq Y$ and $-Y \subseteq S(x)$. Since $R \cup -S$ is reflexive, we obtain xRx or $x(-S)x$. If xRx, then $R(x) \subseteq Y$ implies $x \in Y$. If $x(-S)x$, then $x \notin S(x)$, in particular, $x \notin -Y$.

(7.4): Let $x \in [U]](Y)$. As above, we have $R(x) \cup -S(x) \subseteq Y$. We need to show that $x \in [U]][U]](Y)$, in other words, $x \in [R][U]](Y) \cap [[S]](-[U]](Y))$.

1. $x \in [R][U]](Y)$: Assume not. Then, $R(x) \not\subseteq [U]](Y)$, and so there is some y such that xRy and $y \notin [R](Y) \cap [[S]](-Y)$.
   (a) $y \notin [R](Y)$: Then, $R(y) \not\subseteq Y$, and there is some z such that yRz and $z \notin Y$. This implies that $xRyRz$, and the transitivity of $R \cup -S$ implies that xRz or $x(-S)z$. Since $R(x) \cup -S(x) \subseteq Y$ we obtain $z \in Y$, a contradiction.
   (b) $y \notin [[S]](-Y)$. Then, $-Y \not\subseteq S(y)$, and there is some z such that $z \notin Y$ and $y(-S)z$. Again by transitivity of $R \cup -S$ we obtain $x(R \cup -S)z$ which again by $R(x) \cup -S(x) \subseteq Y$ contradicts $z \notin Y$.

   Thus, $x \in [R][U]](Y)$.

2. $x \in [[S]](-[U]](Y))$: First, note that

$$\begin{aligned} x \in [[S]](-[U]](Y)) &\iff -[U]] \subseteq S(x) \\ &\iff (\forall y)[y \notin [U]](Y) \text{ implies } xSy] \\ &\iff (\forall y)[x(-S)y \text{ implies } y \in [U]](Y)] \\ &\iff (\forall y)[x(-S)y \text{ implies } y \in [R](Y) \cap [[S]](-Y)]. \end{aligned}$$

Thus, let $x(-S)y$.

(a) $y \in [R](Y)$: Assume not; then, there exists some z such that yRz and $z \notin Y$. Thus, $x(-S)yRz$, and the transitivity of $R \cup -S$ implies that $\langle x,z \rangle \in R \cup -S$. It follows that $z \in Y$ by $R(x) \cup -S(x) \subseteq Y$, a contradiction.

(b) $y \in [[S]](-Y)$: Assume not; then, there is some z such that $z \notin Y$ and $y(-S)z$. As in the previous case we obtain $\langle x,z \rangle \in R \cup -S$ which implies $z \in Y$, a contradiction.

It follows that $[U]]$ satisfies (7.4).

(7.5): Let $x \in Y$; we need to show that $x \in [R][U]]^{\partial}(Y) \cap [[S]][U]](-Y)$.

1. $x \in [R][U]]^{\partial}(Y)$: Assume not; then, there is some y such that xRy and $y \notin [U]]^{\partial}(Y)$. The latter implies that $y \notin \langle R \rangle(Y) \cup -[[S]](Y)$, i.e. $R(y) \cap Y = \emptyset$ and $Y \subseteq S(y)$. Since $R \cup -S$ is symmetric, xRy implies yRx. Now, $R(y) \cap Y = \emptyset$ implies $x \notin Y$, a contradiction.

2. $x \in [[S]][U]](-Y)$: Assume not; then, there is some y such that $y \in [U]](-Y)$ and $x(-S)y$. The first condition implies that $R(y) \cap Y = \emptyset$ and $Y \subseteq S(y)$. Since $x \in Y$, we obtain ySx. On the other hand, $x(-S)y$ implies $y(-S)x$, a contradiction.

Thus, the PS–complex algebra of a **KMIA** frame is in **KMIA**. □

This is similar to the situation that S5 is characterized by the class of frames $\langle X,R \rangle$ where R is an equivalence, as well as by the class $\langle X,R \rangle$, where R is the universal relation on X.

The following repreentation theorem now can be shown along the lines of Theorem 6.4:

Theorem 7.5. *1. Each* **KMIA** *frame is embeddable into the canonical frame of its complex algebra.*

2. Each KMIA is embeddable into the complex algebra of its canonical frame.

8 The logic $K^{\sim}$

In 1985, Solomon Passy (under the name Sulejman Tehlikely) [16] presented a sufficiency counterpart $K^{\star}$ to K with the unary modal operator ◫ ("window") and additional axiom and rule

$$\vdash ◫\neg(\varphi \to \psi) \to (◫\neg\varphi \to ◫\neg\psi) \tag{8.1}$$

$$\text{If } \vdash \varphi, \text{ then } \vdash ◫\neg\varphi. \tag{8.2}$$

Let Fml^* be the set of formulas in the language of K^*. The frame semantics is given by relational structures $\langle X, S, v\rangle$ with two binary relations in such a way that for a valuation $v : \mathsf{Fml}^* \to 2^X$ which acts on the Boolean connectives in the usual way, its action with respect to $\boxplus$ is given by

$$x \in v(\boxplus\varphi) \iff v(\varphi) \subseteq S(x), \tag{8.3}$$

which may be interpreted as φ *is sufficient for accessibility from x if and only if* $v(\varphi) \subseteq S(x)$, in other words,

$$\text{Whenever } y \models_v \varphi, \text{ then } xSy.$$

Assuming the usual interpretation of $\Box$, it is easy to see that

$$\langle X, S, v\rangle, x \models \boxplus\varphi \iff \langle X, X^2 \setminus S, v\rangle, x \models \Box\neg\varphi, \tag{8.4}$$

so that axiomatization, completeness etc. of $K^\star$ are reducible to the corresponding properties of K. Thus, K^* has all the positive as well as the negative qualities of K.

Independently of earlier work by Goldblatt [6], van Benthem [18], Humberstone [9] and others, members of the logic group at Sofia University presented a bimodal logic $K^\sim$ which unified the two approaches. Its modal operators are the normal modality $\Box$, the operator $\boxplus$ satisfying (8.1), (8.2) and the condition that the auxiliary operator $[U]\varphi = \Box\varphi \land \boxplus\neg\varphi$ is an S5 modality.

8.1 Frame semantics of $K^\sim$

Frame models[1] have the form $M = \langle X, R, S, v\rangle$ where $S \subseteq R \subseteq X \times X$, and $v : \mathsf{Var} \to 2^X$ is a valuation over the propositional variables which is extended over the Boolean operators in the usual way. With respect to the modal operators, v acts as follows:

$$x \in v(\Box\varphi) \iff R(x) \subseteq v(\varphi), \tag{8.5}$$

$$x \in v(\boxplus\varphi) \iff v(\varphi) \subseteq S(x). \tag{8.6}$$

The *base of a model* $\langle X, R, S, v\rangle$ is the structure $\langle X, R, S\rangle$. Observe that the base of a model of $K^\sim$ is a weak MIA frame. A model $\langle X, R, S, v\rangle$ of $K^\sim$ is called *special* if $R = S$, and we denote it just by $\langle X, R, v\rangle$.

We say that a formula φ *is satisfied in M at $x \in X$ with respect to v*, written as $x \models_v \varphi$, if $x \in v(\varphi)$. φ is called *valid in M*, written as $M \models_v \varphi$ if $x \models_v \varphi$ for all $x \in X$, i.e. if $v(\varphi) = X$. If $\langle W, R, S\rangle$ is the base of a model of $K^\sim$ we say that φ *is true in* $\langle W, R, S\rangle$, written as $\langle W, R, S\rangle \models \varphi$, if $\langle W, R, S, v\rangle \models_v \varphi$ for all valuations based on $\langle W, R, S\rangle$. If φ is true in all models, we write $K^\sim \models \varphi$.

Two models $M = \langle X, R, S, v\rangle$ and $M' = \langle X', R', S', v'\rangle$ of $K^\sim$ are called *modally equivalent* if for all $\varphi \in \mathsf{Fml}^\sim$,

$$M \models \varphi \iff M' \models \varphi. \tag{8.7}$$

Theorem 8.1. *[5]*

1. *$K^\sim$ is sound and complete with respect to its class of frame models.*
2. *If $M = \langle X, R, S, v\rangle$ is a model of $K^\sim$, then, M is modally equivalent to a special model $\underline{M} = \langle \underline{X}, \underline{R}, \underline{v}\rangle$.*

[1] These are called *generalized models* in [5]

8.2 Algebraic semantics of $K^{\sim}$

If $\langle W,R,S\rangle$ is the base of a model of $K^{\sim}$, we consider its complex algebra $\langle 2^W,\langle R\rangle,[[S]]\rangle$. By Lemma 6.3, $\langle 2^W,\langle R\rangle,[[S]]\rangle \in \mathbf{wMIA}$. For a PS–algebra $\langle B,f,g\rangle$, the structure $\langle \mathrm{Ult}(B),R_f,R_g\rangle$ is a base of a model of $K^{\sim}$ if and only if B is a weak MIA, since $S\subseteq R$ in a model of $K^{\sim}$.

Let $T(\mathsf{Var})$ be the term algebra over the language of $K^{\sim}$ with the set Var of variables; in other words, $T(\mathsf{Var})$ is the absolutely free algebra over the type of PS–algebras generated by Var. Thus, each formula $\varphi(p_1,\ldots,p_n)$ of $K^{\sim}$ can be regarded as an element of $T(\mathsf{Var})$.

Lemma 8.2. *Let $M=\langle X,R,S,v\rangle$ be a model of $K^{\sim}$, and $B_v=\{v(\varphi):\varphi\in\mathsf{Fml}^{\sim}\}$.*

1. $\mathfrak{B}_v=\langle B_v,\cap,\cup,\emptyset,X,\langle R\rangle,[[S]]\rangle\in\mathbf{wMIA}$.
2. *If $\mathfrak{B}$ is a subalgebra of $\mathfrak{Cm}_{PS}(M)$ and v is a mapping onto a set of generators of $\mathfrak{B}$, then $\mathfrak{B}=\mathfrak{B}_v$.*

Proof. 1. By definition, the extension of v over $T(\mathsf{Var})$ is a homomorphism $T(\mathsf{Var})\to\mathfrak{Cm}_{PS}(M)$, thus, $\mathfrak{B}_v$ is a subalgebra of $\mathfrak{Cm}_{PS}(M)$. Since $\mathfrak{Cm}_{PS}(M)\in\mathbf{wMIA}$ and $\mathbf{wMIA}$ is a universal class we obtain $\mathfrak{B}_v\in\mathbf{wMIA}$.

2. This follows again from the definition of the extension of v and the fact that v maps Var onto a set of generators. $\square$

The system $\langle M,B_v\rangle$ is an instance of a *general frame* of [20], see also Sections 1.4 and 5.5 of [1], in particular, Example 5.61.

If $\varphi(p_1,\ldots,p_n)$ is a formula, its corresponding term function (as defined in T_1 and T_2) is denoted by $\tau_\varphi^{\mathfrak{B}}(x_1,\ldots,x_n)$. We say that $\varphi(p_1,\ldots,p_n)$ *is valid in* $\mathfrak{B}$, written as $\mathfrak{B}\models\varphi(p_1,\ldots,p_n)$, if $\tau_\varphi^{\mathfrak{B}}(x_1,\ldots,x_n)\approx 1$. In other words, $\mathfrak{B}\models\varphi(p_1,\ldots,p_n)$ if and only if $\tau_\varphi^{\mathfrak{B}}(v(p_1),\ldots,v(p_n))=1$ for all mappings $v:\mathsf{Var}\to B$. If $\mathbf{K}$ is a class of algebras, then we define $\mathbf{K}\models\varphi(p_1,\ldots,p_n)$ if and only if $\mathfrak{B}\models\varphi(p_1,\ldots,p_n)$ for all $\mathfrak{B}\in\mathbf{K}$.

Theorem 8.3. *For all formulas $\varphi(p_1,\ldots,p_n)$, $K^{\sim}\models\varphi(p_1,\ldots,p_n)$ if and only if* $\mathbf{WMIA}\models\varphi(p_1,\ldots,p_n)$.

Proof. "$\Rightarrow$": Suppose that $K^{\sim}\models\varphi(p_1,\ldots,p_n)$, and that $\mathfrak{B}=\langle B,f,g\rangle\in\mathbf{wMIA}$. By Theorem 6.4, we may suppose that $\mathfrak{B}$ is isomorphic to a complex algebra of a weak MIA frame $\langle X,R,S\rangle$. Then, $\langle X,R,S\rangle\models\varphi(p_1,\ldots,p_n)$ implies $v(\varphi(p_1,\ldots,p_n))=X$ for all valuations $v:\mathsf{Fml}^{\sim}\to 2^X$, and therefore, in particular, $\tau_\varphi^{\mathfrak{B}}(v(p_1),\ldots,v(p_n))=1$ for all mappings $v:\mathsf{Var}\to\mathfrak{B}$. It follows that $\mathfrak{Cm}(X)\models\varphi(p_1,\ldots,p_n)$, and therefore, $\mathfrak{B}\models\varphi(p_1,\ldots,p_n)$.

"$\Leftarrow$": Suppose that $\mathbf{wMIA}\models\varphi(p_1,\ldots,p_n)$, and that $\langle X,R,S\rangle$ is a weak MIA frame with full complex algebra $\mathfrak{B}$. Since $\mathfrak{B}\in\mathbf{wMIA}$ and $\mathbf{wMIA}\models\varphi(p_1,\ldots,p_n)$, we have $\tau_\varphi^{\mathfrak{B}}(v(p_1),\ldots,v(p_n))=X$ for all mappings $v:\mathsf{Var}\to\mathfrak{B}$. Since the extension of v over formulas is the term definition of φ this implies $\langle X,R,S,v\rangle\models\varphi(p_1,\ldots,p_n)$. $\square$

Together with Theorem 8.1 we obtain the following algebraic completeness theorem:

Theorem 8.4. *If φ is a formula in $K^{\sim}$, then $K^{\sim}\vdash\varphi$ if and only if* $\boldsymbol{Eq}(\mathbf{wMIA})\models\varphi$.

Theorem 8.5. *Let $\mathfrak{B}=\langle B,f,g\rangle\in\mathbf{wMIA}$. Then, there is some frame$\langle X,R\rangle$ such that $\langle B,f,g\rangle$ and a subalgebra of $\langle 2^X,\langle R\rangle,[[R]]\rangle$ satisfy the same equations.*

Proof. By the Löwenheim – Skolem Theorem we may suppose that B is at most countable, and by Theorem 4.2, we may suppose that B is a subalgebra of $\langle 2^X, \langle R \rangle, [[S]] \rangle$ for some weak MIA frame $\langle X, R, S \rangle$.

Let $T = \{a_n : n \in \mathbb{N}\}$ be a set of generators of B, and define $v : \mathsf{Var} \to T$ by $v(p_n) = a_n$. Since T generates B, the extension $\overline{v}$ of v over the Lindenbaum – Tarski algebra $\mathfrak{L}$ is a surjective homomorphism onto $\mathfrak{B}$.

Consider the model $M = \langle X, R, S, v \rangle$, then, $\mathfrak{B} = \mathfrak{B}_v$ in the sense of Lemma 8.2, and $M \models \varphi$ if and only if $B_v \models \varphi$ for all $\varphi \in \mathsf{Fml}^{\smile}$. Suppose that $M' = \langle X', R', S', v' \rangle$ is modally equivalent to M, where M' is a special frame. Since the theorems of a model correspond to the equational theory of its general frame, it follows that $\mathbf{Eq}(\mathfrak{B}_v) = \mathbf{Eq}(\mathfrak{B}_{v'})$. □

Corollary 8.6. **KMIA** *is the equational class generated by* **CMIA**.

References

[1] Blackburn, P., de Rijke, M., and Venema., Y. (2001). *Modal Logic*. Cambridge University Press.

[2] Burris, S. and Sankappanavar, H. P. (1981). *A Course in Universal Algebra*. Springer-Verlag, New York.

[3] Düntsch, I. and Orłowska, E. (2000). Beyond modalities: Sufficiency and mixed algebras. In Orłowska, E. and Szałas, A., editors, *Relational Methods in Computer Science Applications*, pages 277–299, Heidelberg. Physica Verlag.

[4] Düntsch, I. and Orłowska, E. (2004). Boolean algebras arising from information systems. *Annals of Pure and Applied Logic*, 127:77–98.

[5] Gargov, G., Passy, S., and Tinchev, T. (1987). Modal Environment for Boolean Speculations. In Skordev, D., editor, *Mathematical Logic and Applications*, pages 253–263, New York. Plenum Press.

[6] Goldblatt, R. (1974). Semantic analysis of orthologic. *Journal of Philosophical Logic*, 3:19–35.

[7] Goldblatt, R. (1993). *Mathematics of Modality*. Number 43 in CSLI Lecture Notes. CSLI Publications, Stanford.

[8] Goldblatt, R. (1995). Elementary generation and canonicity for varieties of Boolean algebras with operators. *Algebra Universalis*, 34:551–607.

[9] Humberstone, I. L. (1983). Inaccessible worlds. *Notre Dame Journal of Formal Logic*, 24:346–352.

[10] Jónsson, B. and Tarski, A. (1951). Boolean algebras with operators I. *American Journal of Mathematics*, 73:891–939.

[11] Koppelberg, S. (1989). *General Theory of Boolean Algebras*, volume 1 of *Handbook on Boolean Algebras*. North–Holland.

[12] Orłowska, E., Radzikowska, A., and Rewitzky, I. (2015). *Dualities for Structures of Applied Logics*. College Publications.

[13] Rautenberg, W. (1980). Splitting lattices of logics. *Arch. math. Logik*, 20:155–159.

[14] Sambin, G. and Vaccaro, V. (1988). Topology and duality in modal logic. *Annals of Pure and Applied Logic*, 37(3):249–296.

[15] Tarski, A. (1941). On the calculus of relations. *Journal of Symbolic Logic*, 6:73–89.

[16] Tehlikeli (Passy), S. (1985). An alternative modal logc, internal semantics and external syntax (A philosophical abstract of a mathematical essay). Manuscript.

[17] Vakarelov, D. (1991). Modal logics for knowledge representation systems. *Theoretical Computer Science*, 90:433–456.

[18] van Benthem, J. (1979a). Minimal deontic logics (Abstract). *Bulletin of the Section of Logic*, 8:36–42.

[19] van Benthem, J. (1984). Correspondence theory. In Gabbay, D. and Guenthner, F., editors, *Handbook of Philosophical Logic: Volume III*, pages 325 – 408. Reidel, Dordrecht.

[20] van Benthem, J. F. A. K. (1979b). Canonical modal logics and ultrafilter extensions. *The Journal of Symbolic Logic*, 44(1):1–8.