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Abstract

Discrete dualities are developed for n-potent MTL–algebras and for 2-potent BL–algebras. That is, classes of frames, or relational systems, are defined that serve as dual counterparts to these classes of algebras. The frames defined here are extensions of the frames that were developed for MTL–algebras in [25], [26]; the additional frame conditions required are given here and also the proofs that discrete dualities hold with respect to such frames. The duality also provides an embedding from an n-potent MTL–algebra, or 2-potent BL–algebra, into the complex algebra of its canonical frame, which is a complete algebra in the lattice sense.
Keywords: Non-classical logics, Discrete duality, MTL–algebra, BL–algebra, Residuated lattice, n-potent law

1 Introduction

Discrete duality is a type of duality where a class of abstract relational systems is a dual counterpart to a class of algebras. These relational systems are referred to as ‘frames’ following the terminology of non-classical logics. There is no topology involved in the construction of these frames, so they may be thought of as having a discrete topology and hence the term: discrete duality. Having a discrete duality for an algebraic semantics for a logic often provides a Kripke-style semantics for the logic. In many cases it can also be used to develop filtration and tableau techniques for the logic. Another typical consequence of such a discrete duality in the case of lattice-ordered algebras is that we obtain a method of completing the algebras, i.e., an embedding of algebras into ones that are complete in the lattice sense.

Establishing discrete duality involves the following steps. Given a class of algebras *Alg* (resp., a class of frames *Fr*) we define a class of frames *Fr* (resp., a class of algebras *Alg*). Next, for any algebra $\mathfrak{L}$ from *Alg* we define its ‘canonical frame’ $\mathfrak{Cf}(\mathfrak{L}) \in Fr$ and for each frame $\mathscr{X}$ in *Fr* we define its ‘complex algebra’ $\mathfrak{Cm}(\mathscr{X}) \in Alg$. A duality between *Alg* and *Fr* holds provided that the following facts are provable:

- Every algebra $\mathfrak{L} \in Alg$ is embeddable into the complex algebra of its canonical frame.
- Every frame $\mathscr{X} \in Fr$ is embeddable into the canonical frame of its complex algebra.

Discrete dualities are developed for MTL–algebras in [25], [26] building on the work of [5]. The underlying order structure of MTL–algebras is a distributive lattice and hence the frames associated with these algebras are based on posets as is well known in the duality for distributive lattices [27]. To capture the properties of the operations of a residuated lattice an additional relation is required satisfying the appropriate conditions and hence the MTL–frames are structures of the form $\langle X, \leq, R \rangle$ where R is a ternary relation on X. The canonical frame of an MTL–algebra is the set of prime filters (in the lattice sense) together with the inclusion relation and a canonical form of R determined by the monoid product. The complex algebra of an MTL–frame is the family of upward closed subsets of X with the union and intersection of sets as the lattice operations. The operations of product and residuation are defined in terms of the relation R in such a way that they satisfy all the MTL–algebra axioms. The two discrete representation theorems for the MTL–algebras and MTL–frames hold.

In this paper we give the additional frame conditions needed to characterize the frames of n-potent MTL–algebras (that is, satisfying $x^n = x^{n+1}$) and establish that the discrete duality for MTL–algebras extends to the n-potent case. Thereafter, we consider BL–algebras; in this case there is no additional frame condition that would extend the discrete duality for MTL–algebras to BL–algebras. If such a duality were to exist, it would provide a completion method for BL–algebras, contradicting a result from [22]. In [4] it is shown that the only varieties of BL–algebras admitting completions are the n-potent ones for some $n \geq 1$. This observation, in part, motivated the current research. A complete solution for all the n-potent varieties of BL–algebras is not obtained here, however we do obtain a discrete duality for 2-potent BL–algebras (that is, satisfying $x^2 = x^3$) that extends the discrete duality for MTL-algebras.

2 Preliminaries

If $\langle P, \leq \rangle$ is an ordered set, and $Q \subseteq P$, we let $\uparrow Q \stackrel{\mathrm{df}}{=} \{p \in P : (\exists q)[q \in Q \text{ and } q \leq p]\}$ be the order filter generated by Q. If $Q = \{u\}$, we just write $\uparrow u$ instead of $\uparrow \{u\}$. Note that $\uparrow \emptyset = \emptyset$. A set $Q \subseteq P$ is called $\uparrow$-*closed* if $\uparrow Q = Q$. For undefined concepts in lattice theory we invite the reader to consult [17].

By a *residuated lattice* we mean an algebra $\mathfrak{L} = \langle L, \vee, \wedge, \otimes, \to, 0, 1 \rangle$ such that the reduct $\langle L, \vee, \wedge, 0, 1 \rangle$ is a bounded lattice, $\langle L, \otimes, 1 \rangle$ is a commutative monoid, and $\to$ is the residual of $\otimes$ with respect to the lattice ordering $\leq$, i.e.,

$$(\forall a, b, c \in L)[a \otimes c \leq b \iff c \leq a \to b].$$

Such an algebra is sometimes called a *bounded, integral, commutative residuated lattice* in the literature.

By a *monoidal t-norm based logic–algebra* (MTL–algebra) we mean a residuated lattice $\mathfrak{L} = \langle L, \vee, \wedge, \otimes, \to, 0, 1 \rangle$ in which the *prelinearity* identity holds:

$$(\forall a, b \in L)[(a \to b) \vee (b \to a) = 1].$$

Since its origin in 2001 in [11] the logic MTL has been a subject of extensive study motivated by the facts that it is complete with respect to the class of lattices endowed with left-continuous t-norms and their residua, and that the necessary and sufficient condition for a t-norm to be residuated is left-continuity.

If an MTL–algebra $\mathfrak{L}$ satisfies additionally

$$\text{DIV} \quad (\forall a, b \in L)[a \wedge b = a \otimes (a \to b)] \qquad \text{(Divisibility)}$$

it is called a *BL–algebra*. The class of BL–algebras is the algebraic counterpart of Hajek's basic logic [18, 19]; it is a common generalization of the classes of Gödel algebras, product algebras, and Wajsberg algebras. For recent surveys we invite the reader to consult [16] or [9].

The class of MTL–algebras is a variety which we denote by **MTL**; the class of BL–algebras is also a variety, which we denote by **BL**.

The associativity of $\otimes$ in both MTL–algebras and BL–algebras allows us to write products $a_1 \otimes a_2 \otimes \ldots \otimes a_n$ unambiguously for $a_1, a_2, \ldots, a_n \in L$. For $a \in L$ we shall write $a^1 = a$, and $a^{n+1} = a^n \otimes a$ for $n \geq 1$.

Theorem 2.1. *[11, Proposition 3] Each MTL-algebra is a subdirect product of linearly ordered MTL-algebras.*

A consequence of the above theorem is that an identity holds in the variety of all MTL–algebras if and only if it holds in all linearly ordered MTL–algebras. A further consequence is that the underlying lattice structure of every MTL-algebra is distributive. The following lemma collects some well known properties of MTL–algebras, see e.g. [2, 11, 25].

Lemma 2.2. *Let $\mathfrak{L} = \langle L, \vee, \wedge, \otimes, \to, 0, 1 \rangle$ be an MTL–algebra, and $a, b, c \in L$.*

1. $a \otimes b \leq a$.
2. $b \leq a \to b$.
3. $a \otimes (a \to b) \leq b$.
4. *If $a \leq b$, then $a \otimes c \leq b \otimes c$.*
5. $a \to (b \vee c) = (a \to b) \vee (a \to c)$.
6. $a \otimes (b \vee c) = (a \otimes b) \vee (a \otimes c)$.
7. $a \otimes (b \wedge c) = (a \otimes b) \wedge (a \otimes c)$.
8. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.
9. $a \vee b = ((a \to b) \to b) \wedge ((b \to a) \to a)$.
10. $a \leq ((a \to b) \to b)$.
11. $((a \to b) \to b) = ((b \to a) \to a)$ *if and only if* $((a \to b) \to b) \leq (a \vee b)$.

It is well known that axiom DIV can be expressed in various ways, as in the following lemma. The proofs are straightforward and are left to the reader:

Lemma 2.3. *Let $\mathfrak{L} = \langle L, \vee, \wedge, \otimes, \to, 0, 1\rangle$ be an MTL–algebra. The following are equivalent:*

1. *DIV.*
2. $(\forall a, b \in L)[b \lneq a$ *implies* $a \otimes (a \to b) = b]$.
3. $(\forall a, b \in L)[b \lneq a$ *implies there is some* $c \in L$ *such that* $b = a \otimes c]$.
4. $(\forall a, b \in L)[a \otimes (a \to b) = b \otimes (b \to a)]$.
5. $(\forall a, b \in L)[(a \to b) \vee (b \to a \otimes (a \to b)) = 1]$.

3 Filters in MTL–algebras

Throughout we suppose that $\mathfrak{L} = \langle L, \vee, \wedge, \otimes, \to, 0, 1\rangle$ is an MTL–algebra and $\mathfrak{F}$ is its set of (lattice) filters, that is, $\uparrow$- and $\wedge$-closed nonempty subsets of L. We note that in MTL–algebras, the notion of 'filter' usually refers to a subset that is $\uparrow$-closed and $\otimes$-closed - as in, e.g., [19] and [11]. A filter F is called *proper* if $0 \notin F$, and it is called *prime* if it is proper and for all $a, b \in L$ we have $a \vee b \in F$ implies $a \in F$ or $b \in F$. The set of all prime filters of $\mathfrak{L}$ is denoted by $\mathrm{Prim}(\mathfrak{L})$.

With some abuse of notation, we extend the operator $\otimes$ to subsets of L:

$$\text{If } A, B \subseteq L \text{ then } A \otimes B \stackrel{\text{df}}{=} \{a \otimes b : a \in A, b \in B\}.$$

The associativity of $\otimes$ extends to products of subsets so we may write $A_1 \otimes A_2 \otimes \ldots \otimes A_n$ unambiguously for $A_1, A_2, \ldots, A_n \subseteq L$. For $A \subseteq L$, we write $A^1 = A$ and $A^{n+1} = A^n \otimes A$ for $n \geq 1$.

Lemma 3.1. *Let $F, G, H \in \mathfrak{F}$. Then, $F \otimes G \subseteq H \iff \uparrow (F \otimes G) \subseteq H$.*

Proof. "$\Rightarrow$": Suppose that $F \otimes G \subseteq H$, and let $a \in \uparrow (F \otimes G)$. Then, there are $b \in F, c \in G$ such that $b \otimes c \leq a$. Since $b \otimes c \in H$ by the hypothesis, and H is a filter, we have $a \in H$.

"$\Leftarrow$": Obvious, since $F \otimes G \subseteq \uparrow (F \otimes G)$. □

Lemma 3.2. *Let $F_1, F_2, \ldots, F_n \in \mathfrak{F}$. Then, $\uparrow (F_1 \otimes F_2 \otimes \ldots \otimes F_n) \in \mathfrak{F}$.*

Proof. The case $n = 2$ was proved in [15, Lemma 6.8]. Let $a \in \uparrow (F_1 \otimes F_2 \otimes \ldots \otimes F_n)$. Then, there are $b_i \in F_i$ such that $b_1 \otimes b_2 \ldots \otimes b_n \leq a$. If $a \leq d$, then $b_1 \otimes b_2 \ldots \otimes b_n \leq d$ and $d \in \uparrow (F_1 \otimes F_2 \otimes \ldots \otimes F_n)$. Next, let $c \in \uparrow (F_1 \otimes F_2 \otimes \ldots \otimes F_n)$, and $d_i \in F_i$ such that $d_1 \otimes d_2 \ldots \otimes d_n \leq c$. Now, $b_i \wedge d_i \in F_i$, thus $(b_1 \wedge d_1) \otimes \ldots \otimes (b_n \wedge d_n) \in F_1 \otimes F_2 \otimes \ldots \otimes F_n$. Hence,

$$(b_1 \wedge d_1) \otimes \ldots \otimes (b_n \wedge d_n) \leq (b_1 \otimes b_2 \ldots \otimes b_n) \wedge (d_1 \otimes d_2 \ldots \otimes d_n) \leq a \wedge c,$$

and, therefore, $a \wedge c \in \uparrow (F_1 \otimes F_2 \otimes \ldots \otimes F_n)$. □

Lemma 3.3. *Let $F_1, F_2, \ldots, F_n \in \mathrm{Prim}(\mathfrak{L})$. Then, $\uparrow (F_1 \otimes F_2 \otimes \ldots \otimes F_n) \in \mathrm{Prim}(\mathfrak{L})$.*

Proof. By Lemma 3.2, $\uparrow (F_1 \otimes F_2 \otimes \ldots \otimes F_n)$ is a filter, and all that is left to show is that it is prime. Since $\uparrow (\uparrow (F \otimes G) \otimes H) = \uparrow (F \otimes G \otimes H)$, it is sufficient to consider the case $n = 2$. Let $a \vee b \in \uparrow (F_1 \otimes F_2)$; then, there are $c \in F_1, d \in F_2$ such that $c \otimes d \leq a \vee b$. Since $\mathfrak{L}$ is residuated, we have $d \leq c \to (a \vee b)$, and thus, from Lemma 2.2(5) we obtain $d \leq (c \to a) \vee (c \to b)$. Since $d \in F_2$ and F_2 is prime we have $c \to a \in F_2$ or $c \to b \in F_2$. Suppose, without loss of generality, that $c \to a \in F_2$. Then, $c \otimes (c \to a) \in F_1 \otimes F_2$, and therefore, as $c \otimes (c \to a) \leq a$, we get $c \otimes (c \to a) \leq a \in \uparrow (F_1 \otimes F_2)$. □

If F, G are filters of $\mathfrak{L}$ let $F \to G \overset{\text{df}}{=} \{a : F \otimes \{a\} \subseteq G\}$.

Lemma 3.4. *Let F, G be filters of L. Then, $F \to G$ is a filter or $F \to G = \emptyset$.*

Proof. Let $a \in F \to G$, i.e. $F \otimes \{a\} \subseteq G$. Let $a \leq b$ and $c \in F$; then $c \otimes a \in G$ by the hypothesis, and thus, $c \otimes b \in G$ by the monotony of $\otimes$. Hence, $b \in F \to G$.

Next, let $a, b \in F \to G$. We need to show that $F \otimes \{a \wedge b\} \subseteq G$, so, let $c \in F$. Since $c \otimes a \in G$ and $c \otimes b \in G$ by the hypothesis, we have $(c \otimes a) \wedge (c \otimes b) \in G$ as well. By Lemma 2.2(7) we have $c \otimes (a \wedge b) = (c \otimes a) \wedge (c \otimes b)$, and therefore, $c \otimes (a \wedge b) \in G$. □

The following example shows that $F \to G$ may be empty:

Example 3.5. *Let $L = [0, 1]$, $\langle L, \vee, \wedge, 0, 1 \rangle$ be the unit interval lattice, $\langle L, \otimes, 1 \rangle$ the multiplicative semigroup of the unit interval, and $x \to y \overset{\text{df}}{=} \min\{1, \frac{y}{x}\}$. Then, $\mathfrak{L} \overset{\text{df}}{=} \langle L, \vee, \wedge, \otimes, \to, 0, 1 \rangle$ is a BL–algebra, sometimes called the* Goguen algebra *or* product algebra. *In $\mathfrak{L}$, each $\uparrow$ – closed set is a filter (and vice versa). Let $F \overset{\text{df}}{=} [\frac{1}{2}, 1]$, $H \overset{\text{df}}{=} (\frac{1}{2}, 1]$. If $F \otimes \{a\} \subseteq H$ for some $a \in L$, then, in particular, $\frac{1}{2} \otimes a = \frac{1}{2} \cdot a > \frac{1}{2}$. However, this is only possible if $a > 1$.*

Lemma 3.6. *Let F, G, H be filters of $\mathfrak{L}$. Then, $G \subseteq F \to H \iff F \otimes G \subseteq H$.*

Proof. "$\Rightarrow$": Suppose that $a \in F$, $b \in G$; we need to show that $a \otimes b \in H$. Now, $a \otimes b \in F \otimes \{b\} \subseteq H$, the latter by $b \in G$ and the hypothesis.

"$\Leftarrow$": Assume $F \otimes G \subseteq H$, and let $a \in G$. Then, $F \otimes \{a\} \subseteq F \otimes G \subseteq H$, and thus, $a \in F \to H$. $\square$

Corollary 3.7. *Let F, H be filters of $\mathfrak{L}$. Then, $F \otimes F \subseteq F \to H \iff F \otimes F \otimes F \subseteq H$.*

Lemma 3.8. *[15, Lemma 6.9], [31, Lemma 2.2] Suppose that F, G are filters of $\mathfrak{L}$, and that H is a prime filter of $\mathfrak{L}$ such that $F \otimes G \subseteq H$. Then, there are prime filters F', G' of $\mathfrak{L}$ such that $F \subseteq F'$, $G \subseteq G'$ and $F' \otimes G' \subseteq H$.*

4 Duality for MTL–algebras

Consider a structure $\mathscr{X} = \langle X, \leq, R \rangle$, where X is a nonempty set, $\leq$ is a partial order on X, and R is a ternary relation on X. For $Y, Z \subseteq X$ define

$$Y \otimes_R Z = \{z : (\exists x, y)[x \in Y, y \in Z, \text{ and } R(x, y, z)]\},$$
$$Y \to_R Z = \{x : (\forall y, z)[y \in Y \text{ and } R(x, y, z) \Rightarrow z \in Z]\}.$$

$\mathscr{X}$ is called an *MTL–frame* if it satisfies FMTL_1 – FMTL_6 below for all $x, x', y, y', z, z', u, v, w \in X$. The right hand side of each condition shows the corresponding algebraic property:

FMTL_1 $R(x, y, z)$ and $x' \leq x$ and $y' \leq y$ and $z \leq z' \Rightarrow R(x', y', z')$. Compatibility of $\otimes_R$ with $\subseteq$

FMTL_2 $(\exists u)[R(y, z, u)$ and $R(x, u, t)] \iff (\exists v)[R(x, y, v)$ and $R(v, z, t)]$. Associativity of $\otimes_R$

FMTL_3 $R(x, y, z) \Rightarrow R(y, x, z)$. Commutativity of $\otimes_R$

FMTL_4 $(R(x, y, z)$ and $R(x, v, w)) \Rightarrow (y \leq w$ or $v \leq z)$. Prelinearity

FMTL_5 $(\forall z)(\exists y)[R(z, y, z)]$. $Y \subseteq Y \otimes_R 1$

FMTL_6 $R(x, y, z) \Rightarrow x, y \leq z$. $Y \otimes_R 1 \subseteq Y$

Let $\mathscr{X}$ be an MTL–frame, and let $L(\mathscr{X})$ be the collection of all order filters of $\mathscr{X}$, i.e. $Y \in L(\mathscr{X}) \iff Y = \uparrow Y$. We observe in passing that $\uparrow \emptyset = \emptyset$, and thus, $\emptyset \in L(\mathscr{X})$. The *complex algebra of $\mathscr{X}$* is the algebra

$$\mathfrak{Cm}(\mathscr{X}) \stackrel{\text{df}}{=} \langle L(\mathscr{X}), \cup, \cap, \otimes_R, \to_R, \emptyset, X \rangle.$$

Lemma 4.1. *[25], [26] If $\mathscr{X}$ is an MTL-frame, then $\mathfrak{Cm}(\mathscr{X})$ is an MTL–algebra.*

If $\mathscr{X}$ is an MTL–frame and $Y_1, Y_2, \ldots, Y_n \in L(\mathscr{X})$, then the associativity of $\otimes_R$ allows us to write $Y_1 \otimes_R Y_2 \otimes_R \ldots \otimes_R Y_n$ unambiguously. For $Y \in L(\mathscr{X})$ we write $Y^1 = Y$, and $Y^{n+1} = Y^n \otimes_R Y$ for $n \geq 1$.

The following lemma whose proof follows easily from the definition and associativity of $\otimes_R$ will be helpful later on.

Lemma 4.2. *Let $\mathscr{X} = \langle X, \leq, R \rangle$ be an MTL–frame, $Y \in L(\mathscr{X})$ and $n \geq 2$. Then,*

(4.1)
$$z \in Y^n \iff (\exists y_1, \ldots, y_n, x_1, \ldots, x_{n-1} \in Y)[y_n = z \text{ and } R(y_i, x_i, y_{i+1}), 1 \leq i \lneq n].$$

Let $\mathfrak{L} = \langle L, \vee, \wedge, \otimes, \to, 0, 1 \rangle$ be an MTL–algebra. The *canonical frame* of $\mathfrak{L}$ is the structure

$$\mathfrak{Cf}(\mathfrak{L}) = \langle \mathrm{Prim}(\mathfrak{L}), \subseteq, R_\otimes \rangle$$

where $\mathrm{Prim}(\mathfrak{L})$ is the set of all prime filters of $\mathfrak{L}$ and $R_\otimes$ is the complex relation induced by $\otimes$, i.e., for $F, G, H \in \mathrm{Prim}(\mathfrak{L})$,

$$R_\otimes(F, G, H) \overset{\mathrm{df}}{\iff} F \otimes G \subseteq H. \tag{4.2}$$

In other words,

$$R_\otimes(F, G, H) \iff (\forall a, b)[a \in F \text{ and } b \in G \Rightarrow a \otimes b \in H]. \tag{4.3}$$

Lemma 4.3. *[25], [26] If $\mathfrak{L}$ is an MTL–algebra, then $\mathfrak{Cf}(\mathfrak{L})$ is an MTL–frame.*

Theorem 4.4. *[25], [26][1] Let $\mathfrak{L}$ be an MTL–algebra and $\mathscr{X}$ an MTL–frame.*

1. *$\mathfrak{Cf}(\mathfrak{L})$ is an MTL–frame and $\mathfrak{L}$ can be embedded into the complex algebra of its canonical frame via the mapping $h : L \to \mathfrak{Cm}(\mathfrak{Cf}(\mathfrak{L}))$ defined by $h(a) = \{F \in \mathrm{Prim}(\mathfrak{L}) : a \in F\}$.*
2. *$\mathfrak{Cm}(\mathscr{X})$ is an MTL–algebra and $\mathscr{X}$ can be embedded into the canonical frame of its complex algebra via the mapping $k : X \to \mathfrak{Cf}(\mathfrak{Cm}(\mathscr{X}))$ defined by $k(x) = \{Y \in L(\mathscr{X}) : x \in Y\}$.*

[1] One of the referees pointed out that this follows from a more general result in [5] which also uses a representation with ternary frames.

The complex algebra of an MTL-frame is a *complete* MTL–algebra in the sense that its underlying lattice order is complete, i.e., all infinite meets and joins exist. This is evident from the fact that the meets and joins are intersections and unions, respectively, of $\uparrow$-closed subsets and hence are $\uparrow$-closed themselves. Thus, the above theorem also provides a method of embedding any MTL–algebra into a complete MTL–algebra, namely the complex algebra of its canonical frame.

As **BL** is a subclass of **MTL** one may ask whether there is a corresponding duality theorem for **BL** on the basis of the constructions above, e.g. by adding additional frame conditions. If such additional frame conditions existed, it would imply, as discussed in the previous paragraph, that every BL–algebra can be embedded into a complete BL–algebra. The following example shows that this is not the case (a different example can be found in [22]):

Example 4.5. *Let $\mathfrak{L}$ be the Goguen BL–algebra of Example 3.5. Then $\mathfrak{L}$ is also an MTL–algebra and so $\mathfrak{Cm}(\mathfrak{Cf}(\mathfrak{L}))$ is an MTL–algebra into which $\mathfrak{L}$ embeds by Theorem 4.4. In $\mathfrak{L}$, each proper filter is a prime filter and has the form $(a,1]$ or $[a,1]$ for some $a \in L, a \neq 0$. In particular, $\{1\}$ is a prime filter. Let $F \stackrel{\mathrm{df}}{=} [\frac{1}{2},1]$, and $H \stackrel{\mathrm{df}}{=} (\frac{1}{2},1]$; then, $H \subsetneq F$. Set $Z \stackrel{\mathrm{df}}{=} \uparrow F$ and $Y \stackrel{\mathrm{df}}{=} \uparrow H$, where the $\uparrow$ is taken in the partial order $\langle \mathrm{Prim}(\mathfrak{L}), \subseteq \rangle$; then, Y and Z are increasing sets of prime filters, i.e. $Y,Z \in L(\mathfrak{Cf}(\mathfrak{L}))$, and $Z \subsetneq Y$.*

Assume that $\mathfrak{Cm}(\mathfrak{Cf}(\mathfrak{L}))$ satisfies DIV. Then, $Z \subseteq Y \otimes_R (Y \to_R Z)$, in particular, $F \in Y \otimes_R (Y \to_R Z)$. Thus, there are $G \in Y$, $G' \in Y \to_R Z$ with $G \otimes G' \subseteq F$. Since $G \in Y$ and $Y = \uparrow H$, we have $H \subseteq G$.

The next task is to show that $G' = \{1\}$: Assume there is some $a \in G'$ such that $a \neq 1$. Since $0 < a < 1$, we have $\frac{1}{2} < \frac{1}{2a}$. Choose some x with $\frac{1}{2} < x < \frac{1}{2a}$; then, $x \in H \subseteq G$, and thus, $x \otimes a \in G \otimes G' \subseteq F$. On the other hand, $x \otimes a < \frac{1}{2a} \otimes a = \frac{1}{2}$ and thus, $x \otimes a \notin F$, a contradiction.

By definition of $R_\otimes$ we have $R_\otimes(\{1\},H,H)$, and $\{1\} = G' \in Y \to_R Z$ implies that $H \in Z$, i.e. $F \subseteq H$, a contradiction.

5 n-potent MTL–algebras

Throughout this section, $\mathscr{X} = \langle X, \leq, R \rangle$ is an MTL–frame and $\mathfrak{L} = \langle L, \vee, \wedge, \otimes, \to, 0, 1 \rangle$ an MTL–algebra.

For each integer $n \geq 1$ we define the class of *n-potent MTL–algebras* as the class of MTL–algebras satisfying the identity: $(\forall a)[a^n = a^{n+1}]$.

The 1-potent case is not very interesting as it implies that $\otimes = \wedge$ in all such algebras; in fact, this is the variety of Heyting algebras generated by all linearly ordered Heyting algebras. It is also the variety of Gödel algebras.

The aim of this section is to establish a discrete duality between n-potent MTL–algebras and a special class of MTL–frames. To this end we consider the following frame conditions for $n \geq 2$.

FMTL^n For all $y_1, \ldots, y_n, x_1, \ldots, x_{n-1}$, if $R(y_i, x_i, y_{i+1})$ for all $1 \leq i \leq n-1$, then there exist $v \in \{y_1, x_1, \ldots, x_{n-1}\}$ and $u_1, \ldots, u_{n+1}$ such that $u_1 = v$, $u_{n+1} \leq y_n$ and $R(u_i, v, u_{i+1})$ for all $i \in \{1, \ldots, n\}$.

First, we shall prove some preparatory lemmas:

Lemma 5.1. *Let* $a_1, \ldots, a_n \in L$. *Then,* $a_1 \otimes \ldots \otimes a_n \leq a_1^n \vee \ldots \vee a_n^n$.

Proof. By Theorem 2.1, it suffices to show the claim for linearly ordered $\mathfrak{L}$, in which case the maximum element of $\{a_1, \ldots, a_n\}$ exists, say a_k for some $k \in \{1, \ldots, n\}$. Then, $a_1 \otimes \ldots \otimes a_n \leq a_k^n \leq a_1^n \vee \ldots \vee a_n^n$. $\square$

The n-potent property has a straightforward translation to inclusion of filters:

Lemma 5.2.

$$(\forall F \in \mathfrak{F})[F^{n+1} \subseteq \uparrow (F^n)] \iff (\forall a \in L)[a^n \leq a^{n+1}].$$

Proof. "$\Rightarrow$": Let $a \in L$ and set $F \stackrel{\text{df}}{=} \uparrow \{a\}$. Then, $a^{n+1} \in F^{n+1}$, and by the hypothesis there are $b_1, \ldots, b_n \in F$ such that $b_1 \otimes \ldots \otimes b_n \leq a^{n+1}$. The definition of F implies $a \leq b_i$ for all $i \in \{1, \ldots, n\}$, and from the monotony of $\otimes$ we obtain $a^n \leq b_1 \otimes \ldots \otimes b_n \leq a^{n+1}$.

"$\Leftarrow$": Let $F \in \mathfrak{F}$ and $a_1, \ldots, a_{n+1} \in F$; we need to show that $a_1 \otimes \ldots \otimes a_{n+1} \in \uparrow (F^n)$. Let $p \stackrel{\text{df}}{=} a_1 \wedge \ldots \wedge a_{n+1}$. Since F is a filter, $p \in F$ and, therefore, $p^n \in F^n$. Now,

$$\begin{aligned} p^n &= (a_1 \wedge \ldots \wedge a_{n+1})^n \\ &\leq (a_1 \wedge \ldots \wedge a_{n+1})^{n+1} && \text{by the hypothesis,} \\ &\leq a_1 \otimes \ldots \otimes a_{n+1} && \text{since } \otimes \text{ respects } \leq. \end{aligned}$$

Since $p^n \in F^n$, it follows that $a_1 \otimes \ldots \otimes a_{n+1} \in \uparrow (F^n)$. $\square$

The next result is the key observation for establishing the discrete duality:

Lemma 5.3. *Let $\mathfrak{L}$ be an n-potent MTL–algebra and $F_1,\dots,F_n \in \mathrm{Prim}(\mathfrak{L})$. Then, there exists $i \in \{1,\dots,n\}$ such that $F_i^{n+1} \subseteq \uparrow(F_1 \otimes \dots \otimes F_n)$.*

Proof. Assume that $F_i^{n+1} \not\subseteq \uparrow(F_1 \otimes \dots \otimes F_n)$ for all $i \in \{1,\dots,n\}$. Then, for each $i \in \{1,\dots,n\}$, there are $a_1^i,\dots,a_n^i,a_{n+1}^i \in F_i$ such that $a_1^i \otimes \dots \otimes a_n^i \otimes a_{n+1}^i \notin \uparrow(F_1 \otimes \dots \otimes F_n)$. Set $d_i \overset{\mathrm{df}}{=} a_1^i \wedge \dots \wedge a_n^i \wedge a_{n+1}^i$; then, $d_i \in F_i$. Furthermore,

$$d_i^{n+1} = (a_1^i \wedge \dots \wedge a_n^i \wedge a_{n+1}^i)^{n+1} \leq a_1^i \wedge \dots \wedge a_n^i \wedge a_{n+1}^i \notin \uparrow(F_1 \otimes \dots \otimes F_n),$$

and therefore, $d_i^{n+1} \notin \uparrow(F_1 \otimes \dots \otimes F_n)$. Since $\mathfrak{L}$ is an n-potent MTL–algebra, this implies

$$d_i^n \notin \uparrow(F_1 \otimes \dots \otimes F_n) \tag{5.1}$$

for all $i \in \{1,\dots,n\}$. Now, $d_1 \otimes \dots \otimes d_n \in \uparrow(F_1 \otimes \dots \otimes F_n)$ and, therefore, since $\uparrow(F_1 \otimes \dots \otimes F_n)$ is a filter, by Lemma 3.2, and $d_1 \otimes \dots \otimes d_n \leq d_1^n \vee \dots \vee d_n^n$ by Lemma 5.1, we have $d_1^n \vee \dots \vee d_n^n \in \uparrow(F_1 \otimes \dots \otimes F_n)$. Since $\uparrow(F_1 \otimes \dots \otimes F_n)$ is prime by Lemma 3.3, it follows that $d_i^n \in \uparrow(F_1 \otimes \dots \otimes F_n)$ for some $i \in \{1,\dots,n\}$. This contradicts (5.1). $\square$

Lemma 5.4. *The canonical frame of an n-potent MTL–algebra $\mathfrak{L}$ satisfies FMTL^n.*

Proof. Let $F_1,\dots,F_n,G_1,\dots,G_{n-1} \in \mathrm{Prim}(\mathfrak{L})$ such that $R_\otimes(F_i,G_i,F_{i+1})$, i.e., $F_i \otimes G_i \subseteq F_{i+1}$, for all $i \in \{1,\dots,n-1\}$. Then we have

$$F_1 \otimes G_1 \otimes \dots \otimes G_{n-1} \subseteq F_2 \otimes G_2 \otimes \dots \otimes G_{n-1} \cdots \subseteq F_{n-1} \otimes G_{n-1} \subseteq F_n.$$

By Lemma 5.3, there exists $H \in \{F_1,G_1,\dots,G_{n-1}\}$ such that $H^{n+1} \subseteq \uparrow(F_1 \otimes G_1 \otimes \dots \otimes G_{n-1})$. Since F_n is a filter, we have $\uparrow(F_1 \otimes G_1 \otimes \dots \otimes G_{n-1}) \subseteq F_n$ and so $H^{n+1} \subseteq F_n$. For each $i \in \{1,\dots,n+1\}$ set $U_i \overset{\mathrm{df}}{=} \uparrow(H^i)$. By Lemma 3.3, each U_j is a prime filter. Furthermore, $U_1 = H$, $U_{n+1} \subseteq F_n$ and $U_i \otimes H \subseteq U_{i+1}$, i.e., $R_\otimes(U_i,H,U_{i+1})$, for all $i \in \{1,\dots,n\}$. $\square$

Lemma 5.5. *If $\mathscr{X} = \langle X,\leq,R\rangle$ is an MTL–frame which satisfies FMTL^n, then its complex algebra is an n-potent MTL–algebra.*

Proof. Suppose that $Y \in L(\mathscr{X})$, i.e., Y is an $\uparrow$-closed subset of X, and $n \geq 2$; we shall show that $Y^n \subseteq Y^{n+1}$. Let $z \in Y^n$; then, by Lemma 4.2, there exist $y_1,\dots,y_n, x_1,\dots,x_{n-1} \in Y$ such that $y_n = z$ and $R(y_i,x_i,y_{i+1})$ for all $i \in \{1,\dots,n-$

$1\}$. Thus, the hypothesis of FMTL^n is fulfilled, so there exist $v \in \{y_1, x_1, \ldots, x_{n-1}\}$ and $u_1, \ldots, u_{n+1}$ such that $u_1 = v$, $u_{n+1} \leq z$ and $R(u_i, v, u_{i+1})$ for all $i \in \{1, \ldots, n\}$. Since $v \in Y$ and $u_1 = v$, we have $u_1 \in Y$. By the definition of $\otimes_R$, since $R(u_1, v, u_2)$, we get $u_2 \in Y^2$ and, continuing in this way, we get $u_{n+1} \in Y^{n+1}$. Then $u_{n+1} \leq z$ implies $z \in Y^{n+1}$, since $Y^{n+1} \in L(\mathscr{X})$ and hence is $\uparrow$-closed. $\square$

Using the same mappings as in Theorem 4.4, the previous lemmas allow us to obtain the following duality result:

Theorem 5.6. *Let $\mathfrak{L}$ be an n-potent MTL–algebra and $\mathscr{X}$ an MTL–frame that satisfies FMTL^n.*

1. *$\mathfrak{Cf}(\mathfrak{L})$ is an MTL–frame that satisfies FMTL^n and $\mathfrak{L}$ can be embedded into the complex algebra of its canonical frame.*
2. *$\mathfrak{Cm}(\mathscr{X})$ is an n-potent MTL–algebra and $\mathscr{X}$ can be embedded into the canonical frame of its complex algebra.*

6 2-potent BL–algebras

The following was shown in [4]:

Theorem 6.1. *Suppose that* **V** *is a subvariety of* **BL**. *Then,* **V** *admits completions if and only if the identity $(\forall a)[a^n = a^{n+1}]$ is satisfied for some integer $n \geq 1$.*

The necessary condition for a subvariety of **BL** to have a representation theorem such that the representation algebra is a complete BL–algebra is that it is n-potent for some $n \geq 1$. Thus, a necessary condition for a subvariety of **BL** to have a duality theorem is that it is n-potent for some $n \geq 1$. Note that the Goguen algebra of Example 3.5 does not have this property. In this section we describe a duality theorem for 2-potent BL–algebras.

Recall that a BL–algebra is a residuated lattice which satisfies the prelinearity condition and DIV. Consider the following identity:

$$\mathrm{V}^2 \;:\; (\forall a, b)[(a \to b) \vee (b \to a \otimes b) = 1].$$

Theorem 6.2. *The variety of 2-potent BL–algebras is precisely the variety of residuated lattices that satisfy V^2.*

Proof. The variety of 2-potent BL–algebras is generated by algebras that are ordinal sums of copies of the following two BL–algebras: the two–element Boolean algebra (with $\otimes = \wedge$) and the three–element chain BL-algebra on a $\{0,a,1\}$ with $0 < a < 1$ and $a \otimes a = 0$ (see, e.g., [19], [6]). Observe that the latter is isomorphic to the three element MV-chain.

Since algebras of this form are linearly ordered, for any two elements a,b in such an algebra we have $a \leq b$ or $b \lneq a$. In the first case, $a \to b = 1$. Suppose $b \lneq a$. If a and b are in different components of the ordinal sum, then $a \otimes b = b$. Otherwise, both a and b are in the same two–element or three–element chain, and is easy to check that also $a \otimes b = b$. Hence, $b \to a \otimes b = 1$, so the algebra satisfies V^2. Consequently, the variety of 2-potent BL–algebras satisfies V^2.

Conversely, suppose that $\mathfrak{L} = \langle L, \vee, \wedge, \otimes, \to, 0, 1 \rangle$ is a residuated lattice that satisfies V^2 and let $a,b \in L$. From $a \otimes b \leq a$ we obtain $b \to a \otimes b \leq b \to a$, hence, from V^2 we obtain

$$1 = (a \to b) \vee (b \to a \otimes b) \leq (a \to b) \vee (b \to a). \tag{6.1}$$

Therefore, $(a \to b) \vee (b \to a) = 1$, and thus $\mathfrak{L}$ is an MTL algebra. All that is left to show is that $\mathfrak{L}$ satisfies DIV and $a^2 = a^3$ for all $a \in L$. By Theorem 2.1 and the fact that V^2 is an identity, we may assume w.l.o.g. that $\mathfrak{L}$ is linearly ordered. Since $a \otimes b \leq a \otimes (a \to b)$ we have $b \to a \otimes b \leq b \to a \otimes (a \to b)$. Thus, V^2 implies

$$1 = (a \to b) \vee (b \to a \otimes b) \leq (a \to b) \vee (b \to a \otimes (a \to b)). \tag{6.2}$$

It follows that $(a \to b) \vee (b \to a \otimes (a \to b)) = 1$, hence $\mathfrak{L}$ satisfies DIV by Lemma 2.3.5.

Suppose that $a \in L$. If $a = a^2$, then $a^2 = a^3$, so, suppose that $a^2 \lneq a$. Then, $a \to a^2 \neq 1$ and from V^2 and the fact that $\mathfrak{L}$ is linearly ordered we obtain $a^2 \to a \otimes a^2 = 1$, and it follows that $a^2 \leq a^3$. Since $a^3 \leq a^2$ by Lemma 2.2.1, we have $a^3 = a^2$. $\square$

A structure $\mathscr{X} = \langle X, \leq, R \rangle$, where X is a nonempty set, $\leq$ is a partial order on X and R is a ternary relation on X is called a *residuated lattice frame* if it satisfies FMTL_1 – FMTL_3, FMTL_5, FMTL_6. We define the complex algebra of a residuated lattice frame in analogy to the complex algebra of an MTL–frame. Such a complex algebra is a residuated lattice - this follows from the proof of the corresponding result for MTL–frames in [25], [26], where the FMTL_4 axiom is only used to show prelinearity. Now consider the following frame condition:

FBL^2 : $(\forall x, y, y', z, z')[(R(x,y,z)$ and $R(x,y',z')) \Rightarrow (R(y,y',z')$ or $y' \leq z)]$.

Lemma 6.3. *If $\mathscr{X} = \langle X, \leq, R\rangle$ is a residuated lattice frame that satisfies FBL^2, then its complex algebra satisfies V^2, i.e. it is a 2-potent BL–algebra.*

Proof. Suppose that Y, Z are $\uparrow$–closed subsets of X; we need to show that

$$(Y \to_R Z) \cup (Z \to_R (Y \otimes_R Z)) = X.$$

Assume that there is some $x \in X$ such that $x \notin Y \to_R Z$ and $x \notin Z \to_R (Y \otimes_R Z)$. By definition of $\to_R$, there are $y, z \in X$ such that $R(x, y, z)$, $y \in Y$, and $z \notin Z$. Similarly, there are $y', z' \in X$ such that $R(x, y', z')$, $y' \in Z$, and $z' \notin Y \otimes_R Z$. Since $R(x, y, z)$ and $R(x, y', z')$, by FBL^2 we have either $R(y, y', z')$ or $y' \leq z$. If $R(y, y', z')$, then $y \in Y$ and $y' \in Z$ imply that $z' \in Y \otimes_R Z$, a contradiction. If $y' \leq z$, then $y' \in Z$ and the fact that Z is $\uparrow$-closed imply that $z \in Z$, a contradiction as well. □

Lemma 6.4. *If $\mathfrak{L}$ is a residuated lattice that satisfies V^2, then its canonical frame satisfies FBL^2.*

Proof. Assume that FBL^2 does not hold in $\mathfrak{L}$. Then there are $F, G, G', H, H' \in \mathrm{Prim}(\mathfrak{L})$, such that the following conditions are satisfied, by the definition of $\otimes$ and by Lemma 3.1:

1. $R_\otimes(F, G, H)$, i.e. $\uparrow(F \otimes G) \subseteq H$,
2. $R_\otimes(F, G', H')$, i.e. $\uparrow(F \otimes G') \subseteq H'$,
3. $-R(G, G', H')$, i.e. $G \otimes G' \not\subseteq H'$,
4. $G' \not\subseteq H$.

Then, $G \otimes G' \not\subseteq \uparrow(F \otimes G')$ and $G' \not\subseteq \uparrow(F \otimes G)$. Choose some $a \in G \otimes G'$ with $a \notin \uparrow(F \otimes G')$. Then, there are $b \in G, d \in G'$ with $b \otimes d = a$. Let $c \in G', c \notin \uparrow(F \otimes G)$, and set $e \stackrel{\mathrm{df}}{=} d \wedge c$. Since $c, d \in G'$ and G' is a filter, we obtain $e \in G'$. Furthermore, $b \otimes e \leq a$, and therefore, $b \otimes e \notin \uparrow(F \otimes G')$, since $a \notin \uparrow(F \otimes G')$. Similarly, since $e \leq c$ and $c \notin \uparrow(F \otimes G)$, it follows that $e \notin \uparrow(F \otimes G)$.

Now, since $\mathfrak{L}$ satisfies V^2, we have $(b \to e) \vee (b \to b \otimes e) = 1$, and since F is a prime filter we obtain $b \to e \in F$ or $b \to b \otimes e \in F$. If $b \to e \in F$, then $(b \to e) \otimes b \in F \otimes G$. Since $(b \to e) \otimes b \leq e$ we obtain $e \in \uparrow(F \otimes G)$, a contradiction. On the other hand, if $e \to b \otimes e \in F$, then $(e \to b \otimes e) \otimes e \in F \otimes G'$, and therefore, $(e \to b \otimes e) \otimes e \leq b \otimes e$ implies $b \otimes e \in \uparrow(F \otimes G')$, a contradiction as well. □

Finally, using the same procedure as in the previous cases, we obtain the following duality result:

Theorem 6.5. *Let $\mathfrak{L}$ be a 2-potent BL–algebra and $\mathscr{X}$ a residuated lattice frame that satisfies FBL^2.*

1. $\mathfrak{Cf}(\mathfrak{L})$ *is a residuated lattice frame that satisfies FBL^2 and $\mathfrak{L}$ can be embedded into the complex algebra of its canonical frame.*

2. $\mathfrak{Cm}(\mathscr{X})$ *is a 2-potent BL–algebra and $\mathscr{X}$ can be embedded into the canonical frame of its complex algebra.*

7 Correspondence theory and syntactic aspects

Correspondence theory is well developed for modal logics which require binary relations in their semantic structures. The Sahlqvist theorem provides a syntactic characterization of a class of modal formulas such that the class of frames which validate those formulas is first order definable. However, this is only an existential non-constructive statement, and a concrete frame condition must be discovered. For that purpose a computer system SQEMA [8] was developed which - if it terminates - generates first order frame conditions for the binary relations determining modal operators in a formula. The system is available at www.fmi.uni-sofia.bg/fmi/logic/sqema/. For the correspondences in logics whose operators require ternary relations in the frames such as the product and its residuals in residuated lattices much less is known. One of the possibilities is to apply the algorithm SCAN which is based on a method of elimination of second order quantifiers from formulas of the monadic second order logic. The elimination method was developed in [1] and then it was described and studied in [30] and [13]; see also [24] and [14]. The foundations of the system can be found in [3]. Given a formula of the monadic second order logic, the algorithm computes - provided that it terminates - an equivalent first order formula. It can be used in the correspondence theory for a search of a first order condition for a relation in a Kripke frame corresponding to a property of an operator expressed as a formula in (the language of) the complex algebra of that frame. However, SCAN is usually not applicable to theories based on the monadic second order logic. Applied to the 2-potence property in the complex algebra of a residuated lattice it did not give any meaningful result.

It was pointed out by one of the referees that a frame condition for knotted rules in the context of residuated lattices can be obtained from an algorithm in [28], not yet published at the time of writing this paper (see also [29]). The frame condition FMTL^n presented in Section 5 is somewhat simpler than the one obtained by that algorithm due to the assumptions of prelinearity and distributivity of the underlying lattices. The algorithm in [28] is developed for general residuated lattices which require two-sorted frames for representation theorems in the style of Dedekind-MacNeille (see [32] and [10]), while for distributive lattices two sorts are not needed.

8 Conclusion and outlook

Since its origin in 2001 the logic MTL has been a subject of extensive study motivated by the facts that it is complete with respect to the class of lattices endowed with left-continuous t-norms and their residua, and that the necessary and sufficient condition for a t-norm to be residuated is left-continuity. The Esteva-Godo-Ono hierarchy [12] of substructural and fuzzy logics and their corresponding algebras starts with the full Lambek calculus with exchange and weakening which in the field of fuzzy logic is referred to as a monoidal logic [21]. An algebra $\mathfrak{L}'$ is above an algebra $\mathfrak{L}$ in the hierarchy whenever $\mathfrak{L}'$ is an axiomatic extension of $\mathfrak{L}$. The logics above MTL are:

SMTL = MTL + $a \wedge \neg a = 0$

IMTL = MTL + $\neg\neg a \leq a$

CMTL = G = MTL + $a \leq a \otimes a$

ΠMTL = SMTL + $\neg\neg c \leq (((a \otimes c) \to (b \otimes c)) \to (a \to b))$

BL = MTL + (DIV) $a \wedge b = a \otimes (a \to b)$

Ł = MV = MTL + $a \vee b = (a \to b) \to b$ = IMTL + (DIV)

$\prod$ = ΠMTL + (DIV)

Bool = CMTL + $a \vee \neg a = 1$.

Discrete dualities for algebras SMTL, IMTL, and CMTL are presented in [25]. In CMTL–algebras, obtained from MTL–algebras by endowing them with the contraction axiom $a \leq a \otimes a$, which in MTL–algebras is equivalent to idempotence $a = a \otimes a$, the product coincides with the meet. A discrete duality for ΠMTL algebras has not been approached yet. The result presented in [4] shed a light on the problem of constructing a completion and, in particular, a discrete duality for axiomatic extensions of MTL satisfying the divisibility axiom. It is proved there that any such axiomatic extension admits completions if and only if it satisfies the n-potent law for some $n \geq 2$. In view of that theorem in the present paper we approached the problem of developing a discrete duality for n-potent MTL–algebras and for 2-potent BL–algebras. The dualities we obtained are presented in Section 5 and Section 6, respectively. It follows that based on the theorem on discrete duality for SMTL–algebras we also get a discrete duality for 2-potent SBL–algebras. The corresponding frame axioms are those of 2-potent BL–frame axioms and

$$(\exists y, z \in X)[R(x,y,z) \text{ and } x \leq y].$$

Similarly, based on the theorem on discrete duality for IMTL–algebras we get a discrete duality for 2-potent MV–algebras. The frame axioms are those of 2-potent BL–algebras and

$$(\forall z \in X)(\exists t \in X)[(R(x,z,t) \Rightarrow (\exists u \in X)R(z,y,u)) \Rightarrow y \leq z].$$

Furthermore, the theorem in [4] implies that no completion exists for SBL and MV alone.

A hierarchy of n-contractive MTL–algebras for $n \geq 2$ is studied in [20], however no representation theorems are achieved. Our results of the present paper provide discrete dualities for C_nMTL algebras for all $n \geq 2$ from that hierarchy. Moreover, together with a discrete duality for IMTL–algebras we obtain discrete dualities for all C_nIMTL, $n \geq 2$, in that hierarchy. A discrete duality for MTL–algebras with the WNM axiom has not been approached yet.

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